Physics 137B (Professor Shapiro) Spring 2010

GSI: Tom Griffin

Homework 7 Solutions

1. We have that:

$$\begin{aligned} H'_{n0}(t) &= -q\mathcal{E}(t) < n|x|0 > \\ &= -q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}\delta_{n1} \end{aligned}$$

Thus only a transition to the first excited state is allowed. Putting this into equation 9.17 of the text, we obtain:

$$\begin{aligned} c_1^{(1)} &= -(ih)^{-1}q\mathcal{E}_0\sqrt{\frac{\hbar}{2m\omega}}\int_0^\infty \exp(-t'/\tau + i\omega t')dt' \\ &= (ih)^{-1}q\mathcal{E}_0\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{(i\omega - 1/\tau)} \\ P_{10}^{(1)} = |c_1^{(1)}|^2 &= q^2\mathcal{E}_0^2\frac{1}{2m\omega\hbar}\frac{1}{(\omega^2 + 1/\tau^2)} \end{aligned}$$

2. Choose a coordinate system so that the electric field is in the z direction. Then, by equation 9.17 of the text:

$$\begin{aligned} c_b^{(1)} &= -(ih)^{-1}q\mathcal{E}_0 z_{ba}\int_{-\infty}^\infty (t'^2 + \tau^2)^{-1}\exp(i\omega t')dt' \\ &= -(ih)^{-1}q\mathcal{E}_0 z_{ba}(\pi/\tau)e^{-\omega\tau} \\ P_{ba}^{(1)} = |c_b^{(1)}|^2 &= (h)^{-2}q^2\mathcal{E}_0^2|z_{ba}|^2(\pi/\tau)^2 e^{-2\omega\tau} \end{aligned}$$

Here, $\omega = (E_2 - E_1)/\hbar = \dfrac{3|E_1|}{4\hbar}$, where $E_1 = -13.6eV$ is the ground state energy of hydrogen.

We just have to calculate $z_{ba} =< 21m|z|100 >$. L_z commutes with z and so this will be zero for $m = 1$ and for $m = -1$. So the only nonzero matrix element is for $m = 0$:

$$\begin{aligned}
z_{ba} =< 210|z|100 > &= \int d^3\mathbf{r}[\frac{1}{4\sqrt{2\pi a_\mu^3}}(r/a_\mu)\exp(-r/2a_\mu)\cos\theta](r\cos\theta)[\frac{1}{\sqrt{\pi a_\mu^3}}\exp(-r/a_\mu)] \\
&= \frac{1}{2\sqrt{2}a_\mu^4}\int_0^\infty r^2 dr(r^2)\exp(-3r/2a_\mu)\int_{-1}^1 d(\cos\theta)\cos^2\theta \\
&= \frac{1}{2\sqrt{2}a_\mu^4}\frac{4!}{(-3/2a_\mu)^5}(\frac{2}{3}) \\
&= \frac{128\sqrt{2}a_\mu}{243}
\end{aligned}$$

3. From equation 9.121 of the text, we have:

$$\begin{aligned}
P_{10} &= \frac{q^2}{2m\hbar\omega^3}\left|\int_{-\infty}^\infty \frac{d\mathcal{E}(t)}{dt}\exp(i\omega t)dt\right|^2 \\
&= \frac{q^2}{2m\hbar\omega^3}\left|-\frac{2\mathcal{E}_0}{\tau^2}\int_{-\infty}^\infty t\exp[-(t/\tau)^2]\exp(i\omega t)dt\right|^2 \\
&= \frac{2q^2\mathcal{E}_0^2}{m\hbar\omega^3\tau^4}\left|\int_{-\infty}^\infty t\exp[-(t/\tau)^2+i\omega t]dt\right|^2 \\
&= \frac{2q^2\mathcal{E}_0^2}{m\hbar\omega^3\tau^4}\left|\exp(-\omega^2\tau^2/4)\int_{-\infty}^\infty t\exp[-(t-i\omega\tau^2/2)^2/\tau^2]dt\right|^2 \\
&= \frac{2q^2\mathcal{E}_0^2}{m\hbar\omega^3\tau^4}\left|\exp(-\omega^2\tau^2/4)\sqrt{\pi}\tau i\omega\tau^2/2\right|^2 \\
&= \frac{\pi q^2\mathcal{E}_0^2\tau^2}{2m\hbar\omega}\exp(-\omega^2\tau^2/2)
\end{aligned}$$

4. (a) We apply Fermi's Golden Rule ($W_{ab} = \frac{2\pi}{\hbar}|H'|^2\delta(E_f - E_i)$) to beta decay. The transition rate to a final state with an electron of energy E and momentum $\mathbf{p}$, plus an antineutrino with energy cq and momentum $\mathbf{q}$ is equal to:

$$W_{ab} = \frac{2\pi}{\hbar}|\frac{G_F\mathcal{M}}{V}|^2\delta(E_0 - E - cq)$$

The total transition rate for beta decay is (summing over all final states):

$$
\begin{aligned}
W &= \frac{2\pi}{\hbar}\sum_{\mathbf{p},\mathbf{q}} |\frac{G_F\mathcal{M}}{V}|^2 \delta(E_0 - E - cq) \\
&= \frac{2\pi}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3/V}\frac{d^3\mathbf{q}}{(2\pi\hbar)^3/V}|\frac{G_F\mathcal{M}}{V}|^2\delta(E_0 - E - cq) \qquad \text{(in the continuum limit)} \\
&= \frac{2\pi}{\hbar}|G_F\mathcal{M}|^2 \int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3}\int_0^\infty \frac{4\pi q^2 dq}{(2\pi\hbar)^3}\delta(E_0 - E - cq) \\
&= \frac{2\pi}{\hbar}|G_F\mathcal{M}|^2 \int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3}\frac{4\pi(\frac{E_0-E}{c})^2}{c(2\pi\hbar)^3} \qquad \text{(with the condition, coming from} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{the delta function integral, that } E < E_0) \\
&= \frac{G_F^2|\mathcal{M}|^2}{2\hbar^7c^3\pi^3}\int_0^\infty dp p^2(E_0 - E)^2 \qquad \text{(with the condition that } E < E_0) \\
&= \frac{G_F^2|\mathcal{M}|^2}{2\hbar^7c^3\pi^3}\int_0^{E_0/c} dp p^2(E_0 - pc)^2 \qquad \text{(since } E \approx pc) \\
&= \frac{G_F^2|\mathcal{M}|^2}{2\hbar^7c^3\pi^3}\int_0^{E_0/c} dp(E_0^2p^2 - 2E_0p^3c + p^4c^2) \\
&= \frac{G_F^2|\mathcal{M}|^2}{2\hbar^7c^3\pi^3}(E_0^2p^3/3 - 2E_0p^4/4c + p^5c^2/5)\Big|_{p=0}^{p=E_0/c} \\
&= \frac{G_F^2|\mathcal{M}|^2E_0^5}{60\hbar^7c^6\pi^3} \\
&= \frac{G_F^2|\mathcal{M}|^2Q_0^5}{60\hbar^7c^6\pi^3} \qquad \text{(since } E_0 \approx Q_0 \text{ in the ultrarelativistic limit)}
\end{aligned}
$$

(b) The muon decay rate is $W_\mu = \dfrac{G_F^2Q_0^5}{192\hbar^7c^6\pi^3}$ where $Q_0 \approx m_\mu c^2$. So $\tau_\mu = 1/W_\mu \approx 3\times 10^{-6}s$.