Physics 137B (Professor Shapiro) Spring 2010

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Homework 2 Solutions

1. (a) The eigenvalues of $\mathbf{J}^2 = (\mathbf{S_1} + \mathbf{S_2})^2$ are $j(j+1)\hbar^2$ (equation 6.145 in the text) where j is a non-negative integer or half-integer . Therefore if two electrons (each of spin 1/2) are in a state $|\psi>$ which is an eigenstate of $\mathbf{J}^2$ with eigenvalue $2\hbar^2$, then:

$$\begin{aligned} j(j+1) &= 2 \\ j^2 + j - 2 &= 0 \\ (j-1)(j+2) &= 0 \\ \Rightarrow j &= 1 \end{aligned}$$

Therefore:

$$\begin{aligned} \mathbf{S_1} \cdot \mathbf{S_2}|\psi> &= (\mathbf{J}^2 - \mathbf{S_1}^2 - \mathbf{S_2}^2)/2|\psi> \\ &= (j(j+1) - s_1(s_1+1) - s_2(s_2+1))\hbar^2/2|\psi> \\ &= (1(2) - 1/2(3/2) - 1/2(3/2))\hbar^2/2|\psi> \\ &= \hbar^2/4|\psi> \end{aligned}$$

So $\mathbf{S_1} \cdot \mathbf{S_2}$ has eigenvalue $\hbar^2/4$ for this state.

(b) The two-electron state has j=1, and so it must be a linear combination of the states $|j=1, j_z=1>$, $|j=1, j_z=0>$ and $|j=1, j_z=-1>$. Thus:

$$|\psi>= \alpha|1,1> +\beta|1,0> +\gamma|1,-1>$$

where α, β and γ are complex coefficients satisfying $|\alpha|^2+|\beta|^2+|\gamma|^2 = 1$ and

$$|1,1>= |\uparrow\uparrow> \qquad |1,0>= \frac{1}{\sqrt{2}}(|\uparrow\downarrow> +|\downarrow\uparrow>) \qquad |1,-1>= |\downarrow\downarrow>$$

(equation 6.302 in the text).

Note that $< J_z >=< \psi|J_z|\psi >= |\alpha|^2 - |\gamma|^2$ and so in order to satisfy $< J_z >= 0$ we require $|\alpha| = |\gamma|$. Note that we **don't** require $\alpha = \gamma = 0$ to satisfy $< J_z >= 0$.

$$
\begin{aligned}
< \psi|S_{1x}S_{2x}|\psi > &= < \psi|S_{1x}S_{2x}(\alpha|1,1> +\beta|1,0> +\gamma|1,-1>) \\
&= < \psi|S_{1x}S_{2x}(\alpha|\uparrow\uparrow> +\frac{\beta}{\sqrt{2}}(|\uparrow\downarrow> +|\downarrow\uparrow>) + \gamma|\downarrow\downarrow>) \\
&= < \psi|(\alpha|\downarrow\downarrow> +\frac{\beta}{\sqrt{2}}(|\downarrow\uparrow> +|\uparrow\downarrow>) + \gamma|\uparrow\uparrow>)\hbar^2/4 \\
&= < \psi|(\alpha|1,-1> +\beta|1,0> +\gamma|1,1>)\hbar^2/4 \\
&= (\gamma^*\alpha + |\beta|^2 + \alpha^*\gamma)\hbar^2/4
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
< \psi|S_{1y}S_{2y}|\psi > &= < \psi|S_{1y}S_{2y}(\alpha|1,1> +\beta|1,0> +\gamma|1,-1>) \\
&= < \psi|S_{1y}S_{2y}(\alpha|\uparrow\uparrow> +\frac{\beta}{\sqrt{2}}(|\uparrow\downarrow> +|\downarrow\uparrow>) + \gamma|\downarrow\downarrow>) \\
&= < \psi|(\alpha(i)^2|\downarrow\downarrow> +\frac{\beta}{\sqrt{2}}(i(-i)|\downarrow\uparrow> +(-i)i|\uparrow\downarrow>) \\
&\qquad\qquad +\gamma(-i)^2|\uparrow\uparrow>)\hbar^2/4 \\
&= < \psi|(-\alpha|1,-1> +\beta|1,0> -\gamma|1,1>)\hbar^2/4 \\
&= (-\gamma^*\alpha + |\beta|^2 - \alpha^*\gamma)\hbar^2/4
\end{aligned}
$$

$$
\begin{aligned}
< \psi|S_{1z}S_{2z}|\psi > &= < \psi|S_{1z}S_{2z}(\alpha|1,1> +\beta|1,0> +\gamma|1,-1>) \\
&= < \psi|S_{1z}S_{2z}(\alpha|\uparrow\uparrow> +\frac{\beta}{\sqrt{2}}(|\uparrow\downarrow> +|\downarrow\uparrow>) + \gamma|\downarrow\downarrow>) \\
&= < \psi|(\alpha|\uparrow\uparrow> +\frac{\beta}{\sqrt{2}}(-|\uparrow\downarrow> -|\downarrow\uparrow>) + \gamma(-1)^2|\downarrow\downarrow>)\hbar^2/4 \\
&= < \psi|(\alpha|1,1> -\beta|1,0> +\gamma|1,1>)\hbar^2/4 \\
&= (|\alpha|^2 - |\beta|^2 + |\gamma|^2)\hbar^2/4
\end{aligned}
$$

Therefore:

$$\begin{aligned}
< \psi|\mathbf{S_1} \cdot \mathbf{S_2}|\psi > &= < \psi|S_{1x}S_{2x}|\psi > + < \psi|S_{1y}S_{2y}|\psi > + < \psi|S_{1z}S_{2z}|\psi > \\
&= (\gamma^*\alpha + |\beta|^2 + \alpha^*\gamma - \gamma^*\alpha + |\beta|^2 - \alpha^*\gamma + |\alpha|^2 \\
&\qquad -|\beta|^2 + |\gamma|^2)\hbar^2/4 \\
&= (|\beta|^2 + |\alpha|^2 + |\gamma|^2)\hbar^2/4 \\
&= \hbar^2/4
\end{aligned}$$

2. For an observable that doesn't explicitly depend on time, equation 5.254 of the text gives:

$$\frac{d}{dt} < \mathbf{S} >= \frac{1}{i\hbar} < [\mathbf{S}, H] >$$

The only term in the Hamiltonian that does not commute with $\mathbf{S}$ is $-\gamma(\mathbf{S}\cdot\mathbf{B})$ and so

$$\begin{aligned}
[S_i, H] &= [S_i, -\gamma(\mathbf{S} \cdot \mathbf{B})] \\
&= -\gamma B_j[S_i, S_j] \\
&= -\gamma B_j i\hbar\epsilon_{ijk}S_k \\
&= -\gamma i\hbar(\mathbf{B} \times \mathbf{S})_i \\
&= \gamma i\hbar(\mathbf{S} \times \mathbf{B})_i
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt} < \mathbf{S} > &= \frac{1}{i\hbar} < [\mathbf{S}, H] > \\
&= \frac{1}{i\hbar} < \gamma i\hbar(\mathbf{S} \times \mathbf{B}) > \\
&= \gamma < \mathbf{S} > \times\mathbf{B}
\end{aligned}$$

3. The angular momentum relation $J_z|j, j_z >= j_z\hbar|j, j_z >$ implies that:

$$\begin{aligned}
J_z|\tfrac{3}{2}, \tfrac{3}{2} > &= \tfrac{3}{2}\hbar|\tfrac{3}{2}, \tfrac{3}{2} > \\
J_z|\tfrac{3}{2}, \tfrac{1}{2} > &= \tfrac{1}{2}\hbar|\tfrac{3}{2}, \tfrac{1}{2} > \\
J_z|\tfrac{3}{2}, -\tfrac{1}{2} > &= -\tfrac{1}{2}\hbar|\tfrac{3}{2}, -\tfrac{1}{2} > \\
J_z|\tfrac{3}{2}, -\tfrac{3}{2} > &= -\tfrac{3}{2}\hbar|\tfrac{3}{2}, -\tfrac{3}{2} >
\end{aligned}$$

These results give us the matrix representation of J_z in the basis given by $\{|\frac{3}{2},\frac{3}{2}>,|\frac{3}{2},\frac{1}{2}>,|\frac{3}{2},-\frac{1}{2}>,|\frac{3}{2},-\frac{3}{2}>\}$

$$J_z = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \hbar$$

Similarly, the relation $J_\pm|j,j_z>=\hbar\sqrt{j(j+1)-j_z(j_z\pm 1)}|j,j_z\pm 1>$ implies that:

$$\begin{aligned} J_+|\tfrac{3}{2},\tfrac{3}{2}> &= 0 \\ J_+|\tfrac{3}{2},\tfrac{1}{2}> &= \sqrt{3}\hbar|\tfrac{3}{2},\tfrac{3}{2}> \\ J_+|\tfrac{3}{2},-\tfrac{1}{2}> &= 2\hbar|\tfrac{3}{2},\tfrac{1}{2}> \\ J_+|\tfrac{3}{2},-\tfrac{3}{2}> &= \sqrt{3}\hbar|\tfrac{3}{2},-\tfrac{1}{2}> \end{aligned}$$

These results give us the matrix representation of J_+ in the basis given by $\{|\frac{3}{2},\frac{3}{2}>,|\frac{3}{2},\frac{1}{2}>,|\frac{3}{2},-\frac{1}{2}>,|\frac{3}{2},-\frac{3}{2}>\}$:

$$J_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \hbar$$

From this we can find J_-, J_x and J_y:

$$J_- = (J_+)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \hbar$$

$$J_x = \frac{J_+ + J_-}{2} = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \frac{\hbar}{2}$$

$$J_y = \frac{J_+ - J_-}{2i} = \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \frac{i\hbar}{2}$$

4. When combining a spin $s_1 = 3/2$ particle with a spin $s_2 = 1/2$ particle, the total spin s^{tot} takes values ranging in integer steps from $|s_1 - s_2| = 1$ to $s_1 + s_2 = 2$. Symbolically, we have:

$$\begin{aligned} \tfrac{3}{2} \otimes \tfrac{1}{2} &= 1 \oplus 2 \\ \text{Dimension of Hilbert Space:} \quad 4 \times 2 &= 3 + 5 \end{aligned}$$

We want to construct the states that are eigenstates of $(S^{tot})^2$ and S_z^{tot}, written in terms of the states that are eigenstates of S_{1z} amd S_{2z}.

j=2: We begin by calculating the 5 states corresponding to j=2. Let us start with the state $|s^{tot} = 2, s_z^{tot} = 2 >$. The Clebsch-Gordon coefficients are only nonzero when $s_z^{tot} = s_{1z} + s_{2z}$. Thus the only nonzero term that contribute to the $s_z^{tot} = 2$ state is the one with $s_{1z} = 3/2$ and $s_{2z} = 1/2$. That is,

$$|s^{tot} = 2, s_z^{tot} = 2 >= |s_1 = \tfrac{3}{2}, s_{1z} = \tfrac{3}{2} > |s_2 = \tfrac{1}{2}, s_{2z} = \tfrac{1}{2} >$$

The lowering operator $S_-^{tot} = S1- + S_{2-}$ is then applied to this state to find the rest of the j=2 states (utilising angular momentum relation $J_\pm|j, j_z >= \hbar\sqrt{j(j+1) - j_z(j_z \pm 1)}|j, j_z \pm 1 >$).

$$\begin{aligned} S_-^{tot}|2,2> &= (S_{1-} + S_{2-})|\tfrac{3}{2}, \tfrac{3}{2} > |\tfrac{1}{2}, \tfrac{1}{2} > \\ 2\hbar|2,1> &= \sqrt{3}\hbar|\tfrac{3}{2}, \tfrac{1}{2} > |\tfrac{1}{2}, \tfrac{1}{2} > +\hbar|\tfrac{3}{2}, \tfrac{3}{2} > |\tfrac{1}{2}, -\tfrac{1}{2} > \\ |2,1> &= \tfrac{\sqrt{3}}{2}|\tfrac{3}{2}, \tfrac{1}{2} > |\tfrac{1}{2}, \tfrac{1}{2} > +\tfrac{1}{2}|\tfrac{3}{2}, \tfrac{3}{2} > |\tfrac{1}{2}, -\tfrac{1}{2} > \end{aligned}$$

$$\begin{aligned} S_-^{tot}|2,1> &= (S_{1-} + S_{2-})(\tfrac{\sqrt{3}}{2}|\tfrac{3}{2}, \tfrac{1}{2} > |\tfrac{1}{2}, \tfrac{1}{2} > +\tfrac{1}{2}|\tfrac{3}{2}, \tfrac{3}{2} > |\tfrac{1}{2}, -\tfrac{1}{2} >) \\ \sqrt{6}\hbar|2,0> &= \sqrt{3}\hbar|\tfrac{3}{2}, -\tfrac{1}{2} > |\tfrac{1}{2}, \tfrac{1}{2} > +\tfrac{\sqrt{3}}{2}\hbar|\tfrac{3}{2}, \tfrac{1}{2} > |\tfrac{1}{2}, -\tfrac{1}{2} > \\ &\qquad +\tfrac{\sqrt{3}}{2}\hbar|\tfrac{3}{2}, \tfrac{1}{2} > |\tfrac{1}{2}, -\tfrac{1}{2} > \\ |2,0> &= \tfrac{1}{\sqrt{2}}|\tfrac{3}{2}, -\tfrac{1}{2} > |\tfrac{1}{2}, \tfrac{1}{2} > +\tfrac{1}{\sqrt{2}}|\tfrac{3}{2}, \tfrac{1}{2} > |\tfrac{1}{2}, -\tfrac{1}{2} > \end{aligned}$$

$$
\begin{aligned}
S_-^{tot}|2,0> &= (S_{1-}+S_{2-})(\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},-\tfrac{1}{2}>|\tfrac{1}{2},\tfrac{1}{2}>+\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},\tfrac{1}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>) \\
\sqrt{6}\hbar|2,-1> &= \tfrac{\sqrt{3}}{2}\hbar|\tfrac{3}{2},-\tfrac{3}{2}>|\tfrac{1}{2},\tfrac{1}{2}>+\tfrac{2}{\sqrt{2}}\hbar|\tfrac{3}{2},-\tfrac{1}{2}>|\tfrac{1}{2},-\tfrac{1}{2}> \\
&\qquad +\tfrac{1}{\sqrt{2}}\hbar|\tfrac{3}{2},-\tfrac{1}{2}>|\tfrac{1}{2},-\tfrac{1}{2}> \\
|2,-1> &= \tfrac{1}{2}|\tfrac{3}{2},-\tfrac{3}{2}>|\tfrac{1}{2},\tfrac{1}{2}>+\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},-\tfrac{1}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>
\end{aligned}
$$

$$
\begin{aligned}
S_-^{tot}|2,-1> &= (S_{1-}+S_{2-})(\tfrac{1}{2}|\tfrac{3}{2},-\tfrac{3}{2}>|\tfrac{1}{2},\tfrac{1}{2}>+\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},-\tfrac{1}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>) \\
2\hbar|2,-2> &= \tfrac{1}{2}\hbar|\tfrac{3}{2},-\tfrac{3}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>+\tfrac{3}{2}\hbar|\tfrac{3}{2},-\tfrac{3}{2}>|\tfrac{1}{2},-\tfrac{1}{2}> \\
|2,-2> &= |\tfrac{3}{2},-\tfrac{3}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>
\end{aligned}
$$

j=1: We next calculate the 3 states corresponding to j=1. Let us start with the state $|s^{tot}=1,s_z^{tot}=1>$. The Clebsch-Gordon coefficients are only nonzero when $s_z^{tot}=s_{1z}+s_{2z}$. Thus the only nonzero terms that contribute to the $s_z^{tot}=1$ state are the ones with $s_{1z}=3/2$, $s_{2z}=-1/2$ and $s_{1z}=1/2$, $s_{2z}=1/2$. That is,

$$|s^{tot}=1,s_z^{tot}=1>=\alpha|\tfrac{3}{2},\tfrac{3}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>+\beta|\tfrac{3}{2},\tfrac{1}{2}>|\tfrac{1}{2},\tfrac{1}{2}>$$

for some α and β satisfying $|\alpha|^2+|\beta|^2=1$.
Note that $S_+^{tot}|s^{tot}=1,s_z^{tot}=1>=0$. So applying the S_+^{tot} operator to the state above yields:

$$
\begin{aligned}
0=S_+^{tot}|1,1> &= (S_{1+}+S_{2+})(\alpha|\tfrac{3}{2},\tfrac{3}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>+\beta|\tfrac{3}{2},\tfrac{1}{2}>|\tfrac{1}{2},\tfrac{1}{2}>) \\
&= \alpha|\tfrac{3}{2},\tfrac{3}{2}>|\tfrac{1}{2},\tfrac{1}{2}>+\beta\sqrt{3}|\tfrac{3}{2},\tfrac{3}{2}>|\tfrac{1}{2},\tfrac{1}{2}> \\
&= (\alpha+\beta\sqrt{3})|\tfrac{3}{2},\tfrac{3}{2}>|\tfrac{1}{2},\tfrac{1}{2}>
\end{aligned}
$$

Therefore $\beta=-\frac{\alpha}{\sqrt{3}}$. Normalization then requires that

$$1=|\alpha|^2+|\beta|^2=|\alpha|^2+|\tfrac{-\alpha}{\sqrt{3}}|^2=\tfrac{4|\alpha|^2}{3}$$

Hence we can take $\alpha=\frac{\sqrt{3}}{2}$ and $\beta=-\frac{1}{2}$.

$$|1,1>=\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},\tfrac{3}{2}>|\tfrac{1}{2},-\tfrac{1}{2}>-\tfrac{1}{2}|\tfrac{3}{2},\tfrac{1}{2}>|\tfrac{1}{2},\tfrac{1}{2}>$$

To find the rest of the j=1 states, the lowering operator $S_-^{tot} = S_{1-} + S_{2-}$ is then applied.

$$\begin{aligned}
S_-^{tot}|1,1> &= (S_{1-}+S_{2-})(\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},\tfrac{3}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{1}{2}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}>) \\
\sqrt{2}\hbar|1,0> &= \tfrac{3}{2}\hbar|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\hbar|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}> \\
&\qquad -\tfrac{1}{2}\hbar|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> \\
|1,0> &= \tfrac{1}{\sqrt{2}}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}>
\end{aligned}$$

$$\begin{aligned}
S_-^{tot}|1,0> &= (S_{1-}+S_{2-})(\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}>) \\
\sqrt{2}\hbar|1,-1> &= \sqrt{2}\hbar|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{\sqrt{3}}{\sqrt{2}}\hbar|\tfrac{3}{2},-\tfrac{3}{2}> |\tfrac{1}{2},\tfrac{1}{2}> \\
&\qquad -\tfrac{1}{\sqrt{2}}\hbar|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> \\
|1,-1> &= \tfrac{1}{2}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},-\tfrac{3}{2}> |\tfrac{1}{2},\tfrac{1}{2}>
\end{aligned}$$

Therefore, we have found all eight states:

$$\begin{gathered}
|2,2>= |\tfrac{3}{2},\tfrac{3}{2}> |\tfrac{1}{2},\tfrac{1}{2}> \\
|2,1>= \tfrac{\sqrt{3}}{2}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}> +\tfrac{1}{2}|\tfrac{3}{2},\tfrac{3}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> \\
|2,0>= \tfrac{1}{\sqrt{2}}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}> +\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> \\
|2,-1>= \tfrac{1}{2}|\tfrac{3}{2},-\tfrac{3}{2}> |\tfrac{1}{2},\tfrac{1}{2}> +\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> \\
|2,-2>= |\tfrac{3}{2},-\tfrac{3}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> \\
|1,1>= \tfrac{\sqrt{3}}{2}|\tfrac{3}{2},\tfrac{3}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{1}{2}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}> \\
|1,0>= \tfrac{1}{\sqrt{2}}|\tfrac{3}{2},\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{1}{\sqrt{2}}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},\tfrac{1}{2}> \\
|1,-1>= \tfrac{1}{2}|\tfrac{3}{2},-\tfrac{1}{2}> |\tfrac{1}{2},-\tfrac{1}{2}> -\tfrac{\sqrt{3}}{2}|\tfrac{3}{2},-\tfrac{3}{2}> |\tfrac{1}{2},\tfrac{1}{2}>
\end{gathered}$$

5. (a) From the Clebsch-Gordon table:

$$|1,0> |1,-1>= \tfrac{1}{\sqrt{2}}|2,-1> +\tfrac{1}{\sqrt{2}}|1,-1>$$

So the probability that $j^{tot} = 2$ is $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

(b) From the Clebsch-Gordon table:

$$|\tfrac{3}{2},\tfrac{1}{2}>|1,0>=\sqrt{\tfrac{3}{5}}|\tfrac{5}{2},\tfrac{1}{2}>+\tfrac{1}{\sqrt{15}}|\tfrac{3}{2},\tfrac{1}{2}>-\tfrac{1}{\sqrt{3}}|\tfrac{1}{2},\tfrac{1}{2}>$$

5. The perturbation to the harmonic oscillator is $H' = bx^4 = C(a+a^\dagger)^4$ where $C := b(\frac{\hbar}{2m\omega})^2$. Using the raising and lowering operators ($a|n>=\sqrt{n}|n-1>$ and $a^\dagger|n>=\sqrt{n+1}|n+1>$) we have:

$$\begin{aligned}
(a+a^\dagger)^2|n> &= (a^{\dagger 2}+a^\dagger a+aa^\dagger+a^2)|n> \\
&= \sqrt{(n+1)}\sqrt{(n+2)}|n+2>+\sqrt{n}\sqrt{n}|n>+\sqrt{n+1}\sqrt{n+1}|n> \\
&\qquad +\sqrt{n}\sqrt{n-1}|n-2> \\
&= \sqrt{(n+1)(n+2)}|n+2>+(2n+1)|n>+\sqrt{n(n-1)}|n-2>
\end{aligned}$$

$$\begin{aligned}
<m|(a+a^\dagger)^4|n> &= ((a+a^\dagger)^2)|m>)^\dagger(a+a^\dagger)^2|n> \\
&= \sqrt{(m+1)(m+2)(n+1)(n+2)}\delta_{m+2,n+2}+(2m+1)(2n+1)\delta_{m,n} \\
&\quad +\sqrt{m(m+1)n(n-1)}\delta_{m-2,n-2}+\sqrt{(m+1)(m+2)}(2n+1)\delta_{m+2,n} \\
&\quad +\sqrt{(m+1)(m+2)n(n-1)}\delta_{m+2,n-2} \\
&\quad +(2m+1)\sqrt{(n+1)(n+2)}\delta_{m,n+2} \\
&\quad +(2m+1)\sqrt{n(n-1)}\delta_{m,n-2}+\sqrt{m(m-1)(n+1)(n+2)}\delta_{m-2,n+2} \\
&\quad +\sqrt{m(m-1)}(2n+1)\delta_{m-2,n} \\
&= [(n+1)(n+2)+(2n+1)^2+n(n+1)]\delta_{m,n} \\
&\quad [\sqrt{(n-1)n}(2n+1)+(2n-3)\sqrt{n(n-1)}]\delta_{m,n-2} \\
&\quad +\sqrt{(n-3)(n-2)n(n-1)}\delta_{m,n-4} \\
&\quad +[(2n+5)\sqrt{(n+1)(n+2)}+\sqrt{(n+2)(n+1)}(2n+1)]\delta_{m,n+2} \\
&\quad +\sqrt{(n+4)(n+3)(n+1)(n+2)}\delta_{m,n+4} \\
&= 3(2n^2+2n+1)\delta_{m,n} \\
&\quad 2(2n-1)\sqrt{(n-1)n}\delta_{m,n-2} \\
&\quad +\sqrt{(n-3)(n-2)(n-1)n}\delta_{m,n-4} \\
&\quad +2(2n+3)\sqrt{(n+1)(n+2)}\delta_{m,n+2} \\
&\quad +\sqrt{(n+4)(n+3)(n+2)(n+1)}\delta_{m,n+4}
\end{aligned}$$

Therefore the first order correction to the energy of level n is:

$$\begin{aligned} E_n^{(1)} &= < n|H'|n > \\ &= < n|C(a+a^\dagger)^4|n > \\ &= 3C(2n^2+2n+1) \end{aligned}$$

The first order correction to the eigenstate of level n is:

$$\begin{aligned} \psi_n^{(1)} &= \sum_{m\neq n} \frac{< m|H'|n >}{E_n^{(0)} - E_m^{(0)}} |m > \\ &= \sum_{m\neq n} \frac{< m|C(a+a^\dagger)^4|n >}{(n-m)\hbar\omega} |m > \\ &= \frac{\sqrt{(n-3)(n-2)(n-1)n}C}{4\hbar\omega}|n-4> + \frac{2(2n-1)\sqrt{(n-1)n}C}{2\hbar\omega}|n-2> \\ &\quad -\frac{2(2n+3)\sqrt{(n+1)(n+2)}C}{2\hbar\omega}|n+2> - \frac{\sqrt{(n+4)(n+3)(n+2)(n+1)}C}{4\hbar\omega}|n+4> \end{aligned}$$

In particular, for n=3, we have:

$$E_3^{(1)} = 75C = 75b(\frac{\hbar}{2m\omega})^2$$

$$\begin{aligned} \psi_3^{(1)} &= \tfrac{10\sqrt{6}C}{2\hbar\omega}|1> - \tfrac{36\sqrt{5}C}{2\hbar\omega}|5> - \tfrac{2\sqrt{210}C}{4\hbar\omega}|7> \\ &= \tfrac{5\sqrt{6}b\hbar}{4m^2\omega^3}|1> - \tfrac{9\sqrt{5}b\hbar}{2m^2\omega^3}|5> - \tfrac{\sqrt{210}b\hbar}{8m^2\omega^3}|7> \end{aligned}$$