Physics 137B (Professor Shapiro) Spring 2010

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Homework 11 Solutions

1. (a)

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f(\theta)|^2 \\
&= |a + b\cos\theta|^2 \\
&= |a|^2 + 2Re(ab)\cos\theta + |b|^2\cos^2\theta \\
\sigma &= \int d\Omega(|a|^2 + 2Re(ab)\cos\theta + |b|^2\cos^2\theta) \\
&= 2\pi\int_{-1}^{1} d(\cos\theta)(|a|^2 + 2Re(ab)\cos\theta + |b|^2\cos^2\theta) \\
&= 4\pi(|a|^2 + |b|^2/3)
\end{aligned}$$

(b) The wavefunction must be symmetric under interchange of the two particles (i.e. $\theta \to (\pi - \theta)$), so

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f(\theta) + f(\pi - \theta)|^2 \\
&= |a + b\cos\theta + a + b\cos(\pi - \theta)|^2 \\
&= 4|a|^2 \\
\sigma &= \int d\Omega(4|a|^2) \\
&= 16\pi|a|^2
\end{aligned}$$

(c) We must average over the four possible orthogonal initial states in an

unpolarized beam (three triplet states and one singlet state).

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{3}{4}|f(\theta) - f(\pi-\theta)|^2 + \frac{1}{4}|f(\theta) + f(\pi-\theta)|^2 \\
&= \frac{3}{4}|2b\cos\theta|^2 + \frac{1}{4}|2a|^2 \\
&= 3|b|^2\cos^2\theta + |a|^2 \\
\sigma &= \int d\Omega(3|b|^2\cos^2\theta + |a|^2) \\
&= 2\pi\int_{-1}^{1} d(\cos\theta)(3|b|^2\cos^2\theta + |a|^2) \\
&= 4\pi(|a|^2 + |b|^2)
\end{aligned}$$

(d) Both spins are $+1/2$, so the particles are in the triplet state.

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f(\theta) - f(\pi-\theta)|^2 \\
&= |2b\cos\theta|^2 \\
&= 4|b|^2\cos^2\theta \\
\sigma &= \int d\Omega(4|b|^2\cos^2\theta) \\
&= 2\pi\int_{-1}^{1} d(\cos\theta)(4|b|^2\cos^2\theta) \\
&= 16\pi|b|^2/3
\end{aligned}$$

2. The scattering cross-section in the Born approximation for any process (elastic or inelastic) is given by:

$$\frac{d\sigma}{d\Omega} = (\frac{m}{2\pi\hbar^2})^2\frac{k_f}{k_i}|V_{eff}(q)|^2$$

where $V_{eff}(q) = L^3 < f|V|i >$, $\vec{q} = \vec{k_f} - \vec{k_i}$ and k_f is always such that energy is conserved.

(a) $|i >= |k_i > | \uparrow > |0 >$ where $|k_i >= \frac{e^{i\mathbf{k_i}\cdot\mathbf{r}}}{\sqrt{L^3}}$ and $|f >= |k_f > | \uparrow > |0 >$ where $|k_f >= \frac{e^{i\mathbf{k_f}\cdot\mathbf{r}}}{\sqrt{L^3}}$.

$$
\begin{aligned}
V_{eff}(q) &= L^3 < f|V|i > \\
&= L^3 < k_f| <\uparrow| < 0|2K\sum_j \delta(\vec{r}-\vec{r_j})\vec{S}\cdot\vec{S_j}|k_i > |\uparrow> |0> \\
&= 2KL^3\sum_j < k_f|\delta(\vec{r}-\vec{r_j})|k_i ><\uparrow|\vec{S}|\uparrow> \cdot < 0|\vec{S_j}|0> \\
&= 2K\sum_j \int d^3\vec{r}e^{-i\vec{q}\cdot\vec{r}}\delta(\vec{r}-\vec{r_j}) <\uparrow|\vec{S}|\uparrow> \cdot < 0|\vec{S_j}|0> \\
&= 2K\sum_j e^{-i\vec{q}\cdot\vec{r_j}}(\frac{\hbar}{2}\hat{z}\cdot < 0|\vec{S_j}|0> \\
&= K\hbar\sum_j e^{-i\vec{q}\cdot\vec{r_j}} < 0|S_{jz}|0>
\end{aligned}
$$

and $k_f = k_i$, so:

$$
\frac{d\sigma}{d\Omega} = (\frac{m}{2\pi\hbar^2})^2|K\hbar\sum_j e^{-i\vec{q}\cdot\vec{r_j}} < 0|S_{jz}|0> |^2
$$

(b) $|i >= |k_i > |\uparrow> |0>$ where $|k_i >= \frac{e^{i\vec{k_i}\cdot\vec{r}}}{\sqrt{L^3}}$ and $|f >= |k_f > |\downarrow> |1>$ where $|k_f >= \frac{e^{i\mathbf{k_f}\cdot\mathbf{r}}}{\sqrt{L^3}}$.

$$
\begin{aligned}
V_{eff}(q) &= L^3 < f|V|i > \\
&= L^3 < k_f| <\downarrow| < 1|2K\sum_j \delta(\vec{r}-\vec{r_j})\vec{S}\cdot\vec{S_j}|k_i > |\uparrow> |0> \\
&= 2KL^3\sum_j < k_f|\delta(\vec{r}-\vec{r_j})|k_i ><\downarrow|\vec{S}|\uparrow> \cdot < 1|\vec{S_j}|0> \\
&= 2K\sum_j \int d^3\vec{r}e^{-i\vec{q}\cdot\vec{r}}\delta(\vec{r}-\vec{r_j}) <\downarrow|\vec{S}|\uparrow> \cdot < 1|\vec{S_j}|0> \\
&= 2K\sum_j e^{-i\vec{q}\cdot\vec{r_j}}(\frac{\hbar}{2}(\hat{x}+i\hat{y})\cdot < 1|\vec{S_j}|0> \\
&= K\hbar\sum_j e^{-i\vec{q}\cdot\vec{r_j}} < 0|S_{j+}|0>
\end{aligned}
$$

where $S_{j+} := S_{jx} + iS_{jy}$ and $k_f = \sqrt{k_i^2 - 2m\Delta/\hbar^2}$, with $\Delta := E_1 - E_0$. So:

$$\frac{d\sigma}{d\Omega} = (\frac{m}{2\pi\hbar^2})^2 \frac{\sqrt{k_i^2 - 2m\Delta/\hbar^2}}{k_i} |K\hbar \sum_j e^{-i\vec{q}\cdot\vec{r_j}} < 0|S_{j+}|0 > |^2$$