Physics 137B (Professor Shapiro) Spring 2010

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Homework 10 Solutions

1. (a) s-wave scattering dominates when:

$$\begin{aligned} kR &<< 1 \\ p = \hbar k &<< \hbar/R = \hbar/(r_0 A^{1/3}) = 1.6 \times 10^{-20} kgms^{-1} \\ \text{(that is, } K.E. = p^2/(2m) &<< 0.5MeV) \end{aligned}$$

(b) We have that $k = \sqrt{2mE/\hbar^2} = 2.2 \times 10^{14} m^{-1}$ and $R = r_0(A)^{1/3} = 6.4 \times 10^{-15} m$. For hard sphere potential scaterring, equation 13.82 of the text applies:

$$\begin{aligned} \delta_0 &= -kR = -1.4 \\ \sigma_0 &= \frac{4\pi}{k^2} \sin^2 \delta_0 = 2.5 \times 10^{-28} m^2 \end{aligned}$$

(c) From equation 13.79 of the text:

$$\begin{aligned} \delta_1 &= \tan^{-1}(\frac{j_1(kR)}{n_1(kR)}) = -0.45 \\ \sigma_1 &= \frac{4\pi}{k^2} 3 \sin^2 \delta_1 = 1.5 \times 10^{-28} m^2 \end{aligned}$$

So the p-wave term is approximately 0.6 of the s-wave term. Note the s-wave term does not dominate since kR is not less than 1 here.

2. For this interaction:

$$\begin{aligned} f(\theta) &= \frac{1}{k}(e^{ika}\sin ka + 3ie^{2ika}\cos\theta) \\ &= \frac{1}{k}e^{ika}\sin ka P_0(\cos\theta) + \frac{1}{k}3ie^{2ika}P_1(\cos\theta) \end{aligned}$$

So $f_0 = \frac{1}{k}e^{ika}\sin ka$

(a) $\frac{d\sigma_0}{d\Omega} = |f_0|^2 = \frac{1}{k^2}\sin^2 ka$

(b) $k = \sqrt{2mE/\hbar^2} = 2.5 \times 10^{11} m^{-1}$ and $a = 2 \times 10^{-15} m$ and $\theta = 0$.
So,

$$\begin{aligned} \frac{d\sigma}{d\Omega}\Big|_{\theta=0} = |f(\theta=0)|^2 &= |\frac{1}{k}(e^{ika}\sin ka + 3ie^{2ika})|^2 \\ &= |\frac{1}{k}(\sin ka + 3i\cos(ka) - 3\sin(ka))|^2 \\ &= \frac{1}{k^2}(4\sin^2 ka + 9\cos^2(ka)) \\ &= \frac{1}{k^2}(4 + 5\cos^2(ka)) \\ &= 1.44 \times 10^{-22} m^2 \end{aligned}$$

and therefore
$N = \frac{d\sigma}{d\Omega}\Big|_{\theta=0} \times d\Omega \times F = 1.44 \times 10^{-22} \times 4\pi \times 10^{-3} \times 10^{18} s^{-1} = 1.8 \times 10^{-6} s^{-1}$

3. The differential scattering cross-section for elastic scattering is given by:

$$\frac{d\sigma}{d\Omega} = (\frac{m}{2\pi\hbar^2})^2 |V_{eff}(q)|^2$$

where $V_{eff}(q) = L^3 < f|V|i >$. For this potential:

$$\begin{aligned}
V_{eff}(q) &= L^3 < f|V|i > \\
&= \int d^3r e^{-i\mathbf{k_f}\cdot\mathbf{r}} V_0 e^{-\alpha r} e^{i\mathbf{k_i}\cdot\mathbf{r}} \\
&= \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} V_0 e^{-\alpha r} \\
&= 2\pi \int_0^\infty dr r^2 \int_{-1}^{1} d(\cos\theta) e^{-iqr\cos\theta} V_0 e^{-\alpha r} \\
&= 2\pi \int_0^\infty dr r^2 \frac{(e^{-iqr} - e^{iqr})}{-iqr} V_0 e^{-\alpha r} \\
&= \frac{2\pi i V_0}{q} \int_0^\infty dr r (e^{(-iq-\alpha)r} - e^{(iq-\alpha)r}) \\
&= \frac{2\pi i V_0}{q} [(iq+\alpha)^{-2} - (-iq+\alpha)^{-2}] \\
&= \frac{2\pi i V_0}{q} [\frac{(-iq+\alpha)^2 - (iq+\alpha)^2}{(q^2+\alpha^2)^2}] \\
&= \frac{8\pi V_0 \alpha}{(q^2+\alpha^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= (\frac{m}{2\pi\hbar^2})^2 |\frac{8\pi V_0 \alpha}{(q^2+\alpha^2)^2}|^2 \\
&= (\frac{4m\alpha V_0}{\hbar^2 (q^2+\alpha^2)^2})^2
\end{aligned}$$

Since we have elastic scattering, $k_f = k_i$, and if we choose our coordinate system so that k_i lies along the z-axis, then

$q^2 = |k_f - k_i|^2 = k_f^2 + k_i^2 - 2k_f k_i \cos\theta = 2k_i^2(1-\cos\theta)$ Therefore the total cross section is:

$$\begin{aligned}
\sigma = \int d\Omega \frac{d\sigma}{d\Omega} &= 2\pi \int_{-1}^{1} d(\cos\theta) (\frac{4m\alpha V_0}{\hbar^2 (2k_i^2(1-\cos\theta) + \alpha^2)^2})^2 \\
&= 2\pi (\frac{4m\alpha V_0}{\hbar^2})^2 \int_{-1}^{1} du (2k_i^2 + \alpha^2 - 2k_i^2 u)^{-4} \\
&= 2\pi (\frac{4m\alpha V_0}{\hbar^2})^2 \Big[\frac{(2k_i^2 + \alpha^2 - 2k_i^2 u)^{-3}}{6k_i^2}\Big]_{u=-1}^{u=1} \\
&= 2\pi (\frac{4m\alpha V_0}{\hbar^2})^2 \frac{1}{6k_i^2} \Big[\frac{1}{\alpha^6} - \frac{1}{(4k_i^2+\alpha^2)^3}\Big]
\end{aligned}$$

4. The differential scattering cross-section for elastic scattering is given by:

$$\frac{d\sigma}{d\Omega} = (\frac{m}{2\pi\hbar^2})^2|V_{eff}(q)|^2$$

where $V_{eff}(q) = L^3 < f|V|i >$. For this potential:

$$\begin{aligned}
V_{eff}(q) &= L^3 < f|V|i > \\
&= \int d^3r e^{-i\mathbf{k_f}\cdot\mathbf{r}} V_0 e^{-\alpha^2 r^2} e^{i\mathbf{k_i}\cdot\mathbf{r}} \\
&= \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} V_0 e^{-\alpha^2 r^2} \\
&= V_0(\int_{-\infty}^{\infty} dx e^{-iq_x x-\alpha^2 x^2})(\int_{-\infty}^{\infty} dy e^{-iq_y y-\alpha^2 y^2})(\int_{-\infty}^{\infty} dz e^{-iq_z z-\alpha^2 z^2}) \\
&= V_0(\frac{\sqrt{\pi}}{\alpha} e^{-q_x^2/4\alpha^2})(\frac{\sqrt{\pi}}{\alpha} e^{-q_y^2/4\alpha^2})(\frac{\sqrt{\pi}}{\alpha} e^{-q_z^2/4\alpha^2}) \\
&= V_0(\frac{\sqrt{\pi}}{\alpha})^3 e^{-q^2/4\alpha^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= (\frac{m}{2\pi\hbar^2})^2|V_0(\frac{\sqrt{\pi}}{\alpha})^3 e^{-q^2/4\alpha^2}|^2 \\
&= \frac{m^2\pi V_0^2 e^{-q^2/2\alpha^2}}{4\hbar^4\alpha^6} \\
&= \frac{m^2\pi V_0^2 e^{-k_i^2(1-\cos\theta)/\alpha^2}}{4\hbar^4\alpha^6}
\end{aligned}$$

Therefore the total cross section is:

$$\begin{aligned}
\sigma = \int d\Omega \frac{d\sigma}{d\Omega} &= 2\pi \int_{-1}^{1} d(\cos\theta) \frac{m^2\pi V_0^2 e^{-k_i^2(1-\cos\theta)/\alpha^2}}{4\hbar^4\alpha^6} \\
&= 2\pi \frac{m^2\pi V_0^2 e^{-k_i^2/\alpha^2}}{4\hbar^4\alpha^6} (\frac{e^{k_i^2/\alpha^2} - e^{-k_i^2/\alpha^2}}{k_i^2/\alpha^2}) \\
&= \frac{m^2\pi^2 V_0^2}{2\hbar^4 k_i^2\alpha^4}(1 - e^{-2k_i^2/\alpha^2})
\end{aligned}$$

5. Equation 13.13 states:

$$\tan\theta_L = \frac{\sin\theta}{\cos\theta + \tau}$$

and therefore:

$$\cos\theta_L = \frac{\cos\theta + \tau}{\sqrt{\sin^2\theta + (\cos\theta + \tau)^2}} = \frac{\cos\theta + \tau}{\sqrt{1 + \tau^2 + 2\tau\cos\theta}}$$

$$\begin{aligned}\frac{d(\cos\theta_L)}{d(\cos\theta)} &= \frac{1}{\sqrt{1 + \tau^2 + 2\tau\cos\theta}} - \frac{(\cos\theta + \tau)\tau}{(1 + \tau^2 + 2\tau\cos\theta)^{3/2}} \\ &= \frac{1 + \tau\cos\theta}{(1 + \tau^2 + 2\tau\cos\theta)^{3/2}}\end{aligned}$$

Therefore,

$$\left(\frac{d\sigma}{d\Omega}\right)_L = \left|\frac{d(\cos\theta_L)}{d(\cos\theta)}\right|\left(\frac{d\sigma}{d\Omega}\right) = \frac{(1 + \tau^2 + 2\tau\cos\theta)^{3/2}}{|1 + \tau\cos\theta|}\left(\frac{d\sigma}{d\Omega}\right)$$