Solutions to Physics 226: Problem Set #2

1. *Central Limit Theorem*

   (a) For the mean:

$$
\begin{aligned}
\mu &\equiv \int_{\infty}^{\infty} x f(x) dx \\
&= \int_0^1 \frac{x}{b-a} dx \\
&= \frac{1}{2} \frac{x^2}{b-a} \Big|_a^b \\
&= \frac{1}{2} \left(\frac{b^2 - a^2}{b-a} \right) \\
&= \frac{a+b}{2}
\end{aligned}
$$

   For the variance:

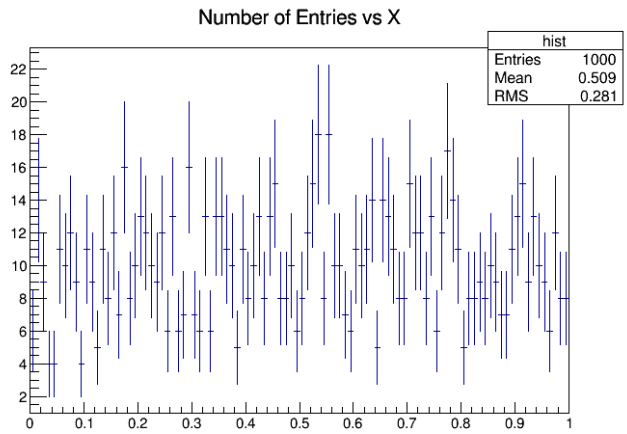
$$
\begin{aligned}
\sigma^2 &\equiv \int_{\infty}^{\infty} x^2 f(x) dx - \mu^2 \\
&= \int_0^1 x^2 dx - \frac{1}{4} \left(\frac{b^2 - a^2}{b-a} \right)^2 \\
&= \frac{1}{3} x^3 \Big|_a^b - \frac{1}{4} (a+b)^2 \\
&= \frac{1}{3} \left(\frac{b^3 - a^3}{b-a} \right) - \frac{1}{4} (a+b)^2 \\
&= \frac{a^2}{3} + \frac{ab}{3} + \frac{b^2}{3} - \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4} \\
&= \frac{1}{12} \left(a^2 - 2ab + b^2 \right) \\
&= \frac{1}{12} (a-b)^2
\end{aligned}
$$

   (b) The root macro I used to solve this part is here:

   http://physics.lbl.gov/shapiro/Physics226/centralLimit.C

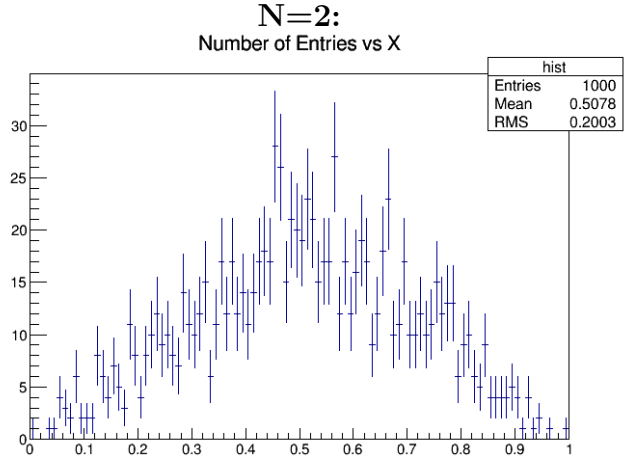
   I found $\mu = 0.509$ and $\sigma = 0.281$. These values are consistent with the calculation of Part (a).

Here is the distribution of random numbers, together with the mean and sigma

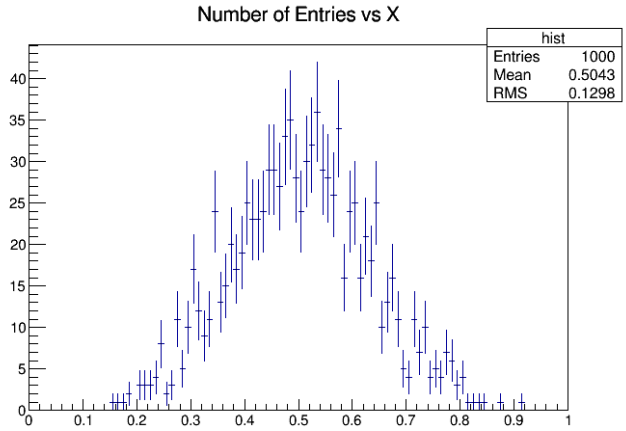


Here is the distribution of S, together with the mean and sigma for $N = 2$, 5 and 10. The returned $\sigma = 1/\sqrt{12N}$ as expected.

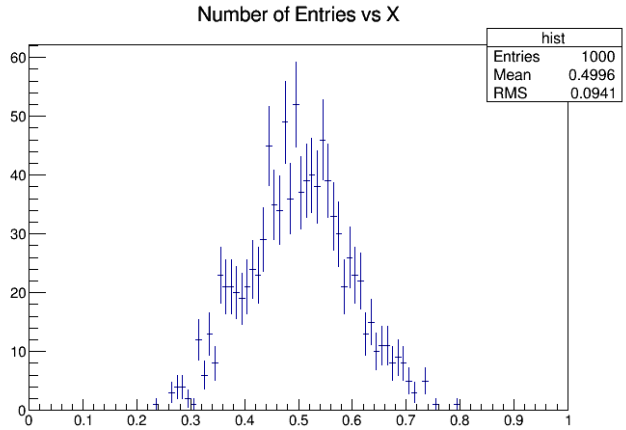
N=2:



N=5:



N=10:



2. *Resolution of a Silicon Detector*

   (a) This is essentially the same problem as above. The expression for the variance is

$$\sigma^2 = \int (x - x_{meas})^2 f(x) dx$$

   where $f(x)$ is the probability distribution function for x. If we let $x_{true} = x$ range from $-\ell/2$ to $\ell/2$ then x_{meas} always is 0 and $f(x) = 1/\ell$. Thus,

$$\sigma^2 = \int_{-\ell/2}^{\ell/2} x^2 (1/\ell) dx = \frac{\ell^2}{12}$$

   Thus $\sigma = \ell/\sqrt{12}$.

(b) The only difference from part (a) is that $x_{meas} = 0$ for the inner two thirds of the strip and $x_{meas} = \ell/2$ for the left sixth and $-\ell/2$ for the right sixth. So

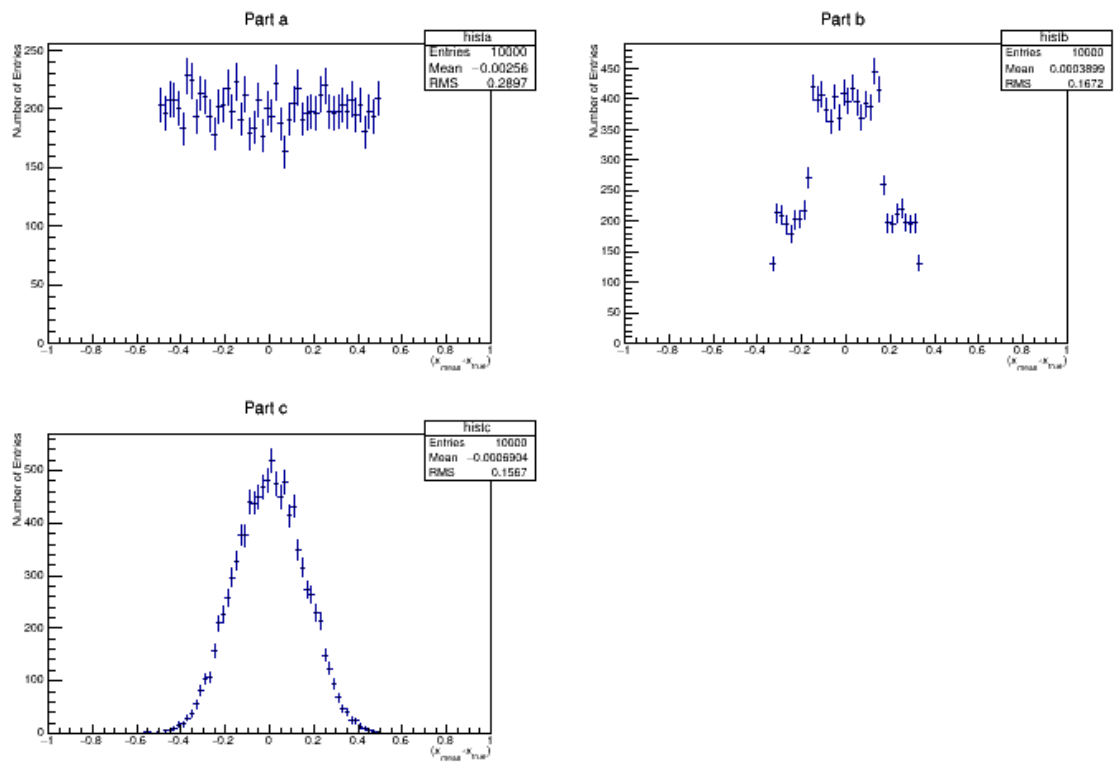
$$\sigma^2 = \int_{-\ell/2}^{-\ell/3} (x-(-\ell/2))^2(1/\ell)dx + \int_{-\ell/3}^{\ell/3} x^2(1/\ell)dx + \int_{\ell/3}^{\ell/2} (x-\ell/2)^2(1/\ell)dx = \frac{\ell^2}{36}$$

Thus $\sigma = \ell/6$

3. The root macro used to solve this problem can be found here:

http://physics.lbl.gov/shapiro/Physics226/siliconResolution.C

Here are the resulting distributions:



Note that for part (a) the RMS of 0.2897 is consistent with the prediction of $1/\sqrt{12} = 0.2887$ and for part (b) the RMS of 0.1672 is consistent with the prediction of $1/6 = 0.1667$. The resolution for part (c) is slightly better than for part (b).