1. Deep Inelastic Scattering Kinematics Consider the process \( ep \rightarrow e + X \). We are going to derive the expression quoted in class that gives the cross section in terms of the structure functions and the definitions of \( x \) and \( y \):

\[
\begin{align*}
    x & \equiv - \frac{q^2}{2p \cdot q} \\
    y & \equiv \frac{p \cdot q}{p \cdot k}
\end{align*}
\]

where \( p \) is the initial four-momentum of the proton, \( k \) is the initial four-momentum of the electron, \( k' \) is the final four-momentum of the electron and \( q \equiv k - k' \).

(a) From the definition of \( y \), show that

\[
1 - y = \frac{p \cdot k'}{p \cdot k} = \frac{1}{2} (1 + \cos \theta^*)
\]

where \( \theta^* \) is the scattering angle in the e-parton center-of-mass frame.

(b) Starting with the expression for e-parton elastic scattering in the center-of-mass frame:

\[
\frac{d\sigma^{eq}}{d\Omega} = e_q^2 \frac{\alpha^2}{8p^2 \sin^4(\theta^*/2)} \left[ 1 + \cos^4(\theta^*/2) \right]
\]

where \( e_q \) is the charge of the quark or antiquark in units of \( e \) (e.g. \( e_q = 2/3 \) for up quarks) and \( p \) is the incoming momentum of the \( e \) in the center-of-mass frame, perform a change of variables to find the differential cross section \( d\sigma^{eq}/dy \).

(c) Use the fact that the deep inelastic \( ep \) cross section can be calculated as an incoherent sum over e-parton scattering cross sections to turn your expression from part (b) into an expression for the \( ep \) scattering cross section \( d^2\sigma^{ep}/dxdy \) in terms of a sum over \( f_i(x) \), where \( f_i(x) \) is the parton distribution function for parton species \( i \).
(d) Use the definition of $F_2(x)$:

$$F_2(x) \equiv \sum_i e_i^2 x f_i(x)$$

to rewrite $d^2\sigma^{ep}/dx dy$ in terms of $F_2(x)$ instead of the sum over the $f_i(x)$.

2. **Parton Model Kinematics** At large momentum transfer, hadron-hadron scattering can be described using the parton model. An old, but very complete, discussion of this process can be found in Reviews of Modern Physics 56: 579707 (SuperCollider physics). We will use the notation of that article here. The cross section for the reaction $a + c \rightarrow c + X$ is given by

$$d\sigma(a + b \rightarrow c + X) = \sum_{ij} f_i^{(a)}(x_a) f_j^{(b)}(x_b) d\hat{\sigma}(i + j \rightarrow c + X)$$

where $f_i(x)$ is the parton distribution function for partons of species $i$ to carry a fraction $x$ of the proton’s momentum (this is the same $f_i(x)$ as in the previous problem) and $d\hat{\sigma}(i + j \rightarrow c + X)$ is the hard scattering cross section. Prove the following:

(a) The invariant mass squared of the hard scattering system $\hat{s} = \tau s$ where $s$ is the center-of-mass energy squared of the hadron-hadron collision and $\tau = x_a x_b$

(b) The longitudinal momentum of the hard scattering system (ie the momentum along the beamline) is $p = x\sqrt{s}/2$ where $x = x_a - x_b$. Our convention is that particle $a$ comes from the left and particle $b$ from the right with $p_{||a} = x_a \sqrt{s}/2$ and $p_{||b} = x_b \sqrt{s}/2$.

(c) The kinematic variables $x_a$ and $x_b$ are related to the variables of the hadronic process by

$$x_{a,b} = \frac{1}{2} \left[ (x^2 + 4\tau)^{\frac{1}{2}} \pm x \right]$$