Physics 226: Problem Set #3
Due in Class on Thurs Sept 22, 2016

1. Consider the problem of determining the lifetime of a species of particle that we can stop in our detector by observing it decays in the presence of a constant background. The rate is given by

$$R(t) = A + Be^{-\Gamma t}$$

We'll take as the true parameters $A = 1/\text{sec}$, $B = 2/\text{sec}$, and $\Gamma = 2/\text{sec}$. Each time we stop a particle in our detector, we will only wait to see if it decays for a maximum time $t_{max} = 3$ seconds. We can imagine doing the experiment over many times (each experiment takes three seconds) to accumulate a lot of data.

   (a) First, let's generate some fake data. We will generate this data using the Monte Carlo technique known as the "acceptance-rejection method" or "rejection sampling." Suppose you want to generate events whose distribution follows the function $f(x)$ (here $f(x)$ is the probability distribution function). This can be achieved by generating points (x_i, y_i) randomly in a two-dimensional space and keeping only the subset of the events where $y_i \leq f(x_i)$. A more detailed explaination of the method can be found here:

   https://en.wikipedia.org/wiki/Rejection_sampling

   We will generate fake data in the following way. The maximum rate is at $t = 0$, where it is $R_{max} = (A + B)$. Pick an x_i, the time t of the decay, randomly between 0 and t_{max}, and pick y_i randomly between 0 and R_{max}. Use $R(t)$ to you decide whether to keep this event. Generate a reasonable amount of such fake data. What percentage of the time do you expect to keep an event? Is that what you find?

   (b) Calculate the negative log-likelihood function, $-\ln \mathcal{L}$, for your data taking care to use a PDF normalized so

$$\int_0^{t_{max}} f(t)dt = 1$$

Using this condition you can express your likelihood in terms of two free parameters, Γ and $\kappa = A/B$. (Note: you can see an example of how this works in the handout *Some Comments on Likelihood Functions* that is posted on our bCourses page).

We will study the simulated data, pretending that we don't know what values of A, B and Γ were used to generate it. We want to determine Γ from the data. Write code to calculate the negative log-likelihood:

$$-\ln \mathcal{L} = -\sum_i \ln f(\kappa, \Gamma, t_i)$$

where the t_i are the time values you accepted in the Monte Carlo process.

(c) There are lots of algorithms for finding the minimum of a non-linear function such as our negative log-likelihood, but we won't bother to use any of these algorithms here. Instead, we will explore the minimum by inspecting the behaviour of the function. Plot (or just show the values in a 2-dimensional grid) the value of $-\ln \mathcal{L}$ you obtain from your simulated data as you vary κ and Γ in the region of the true answer ($\kappa = 0.5$, $\Gamma = 2$). How close is the Γ that gives minimum negative log-likelihood to the true value of Γ? Now do the same, but with ten times as much simulated data. How close is the maximum now to the true value?

(d) We saw in class that for high statistics $-2 \ln \mathcal{L}$ is distributed like a χ^2 distribution and the uncertainty on the estimate of a parameter of the function can be obtained by finding how much the you can change the parameter to increase $-2 \ln \mathcal{L}$ by 1. Assuming you know κ exactly, plot $-2 \ln \mathcal{L}$ for values of Γ around the maximum likelihood and indicate the uncertainty on the measured value of Γ.

Try running your code with different numbers of events per pseudoexperiment and running many pseudoexperiments. Is it true that 68% of the time the true value lies within $\Delta\mathcal{L} = 1$?

2. Using the explicit matrix form of the 8 λ matrices that form the fundamental representation of SU(3), construct the combinations that represent the following

(a) I_3 (the third component of isospin)

(b) U^+ (the operator that increases the 3rd component of U-spin by 1 unit

(c) $\mathbf{V}^2$ (the operator that returns the eigenvalue of V-spin)

(d) The charge operator $\mathcal{Q}$ where $\mathcal{Q}\psi = q\psi$ (q is the charge of the quark and ψ is the quark wave function).