• Quark model says $p$ consists of 3 quarks
  ▶ Are they real?
• Gyromagnetic moment $g_p = 5.586$ is far from the Dirac value of 2 that holds for pointlike spin-$\frac{1}{2}$ particles
  ▶ Pattern of baryon magnetic moments can be explained using quark model with fraction charges, fitting for quark masses
• Size of nucleus consistent with nucleons of size $\sim 0.8$ fm

To study structure of the proton, will use scattering techniques
  Similar idea to Rutherford’s initial discover of the nucleus
Scattering of Spinless Pointlike Particles

- Rutherford Scattering (spinless electron scattering from a static point charge) in lab frame:

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \left(\frac{1}{2} \theta\right)}
\]

Here, \( E \) is energy of incident electron and \( \theta \) is scattering angle in the lab frame.

- Mott Scattering: Taking into account statistics of identical spinless particles

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2 \left(\frac{1}{2} \theta\right)}{4E^2 \sin^4 \left(\frac{1}{2} \theta\right)}
\]
Scattering of Spin-$\frac{1}{2}$ Pointlike Particles

- Elastic Scattering of a spin-$\frac{1}{2}$ electron from a pointlike spin-$\frac{1}{2}$ particle of mass $M$:
  - Elastic scattering of electron from infinite mass target changes angle but not energy
  - For target of finite mass $M$, final electron energy is
    \[ E' = \frac{E}{1 + \frac{2E}{M} \sin^2 \left( \frac{1}{2} \theta \right)} \]
    and the four-momentum transfer is
    \[ q^2 = -4EE' \sin^2 \left( \frac{1}{2} \theta \right) \]

The elastic scattering cross section is:

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2 \left( \frac{1}{2} \theta \right)}{4E^2 \sin^4 \left( \frac{1}{2} \theta \right)} \frac{E'}{E} \left[ 1 - \frac{q^2}{2M^2} \tan^2 \left( \frac{1}{2} \theta \right) \right] \]
What Happens if the Target Particles Have Finite Size?

- Charge distribution $\rho(r)$: $\int \rho(r) d^3r = 1$
- Scattering amplitude modified by a “Form Factor”
  \[ F(q^2) = \int d^3r e^{i\vec{q} \cdot \vec{r}} \rho(r) \]

  So that the cross section is modified by a factor of $|F(q^2)|^2$

- Note: As $q^2 \to 0$, $F(q^2) \to 1$
- We therefore can Taylor expand
  \[ F(q^2) = \int d^3r \left( 1 + i\vec{q} \cdot \vec{r} - \frac{1}{2}(\vec{q} \cdot \vec{r})^2 + \ldots \right) \rho(r) \]
• The first $\vec{q} \cdot \vec{r}$ term vanishes when we integrate

\[
F(q^2) = 1 - \frac{1}{2} \int r^2 dr d\cos \theta d\phi \, \rho(r)(qr)^2 \cos^2 \theta
\]

\[
= 1 - \frac{2\pi}{2} \int dr d\cos \theta \, q^2 r^4 \cos^2 \theta
\]

\[
= 1 - \frac{<r^2>}{4} q^2 \int \cos^2 \theta \, d\cos \theta
\]

\[
= 1 - \frac{<r^2>}{4} q^2 \left[ \cos^3 \theta \right]_0^1
\]

\[
= 1 - \frac{<r^2>}{6} q^2
\]

• For elastic scattering, can relate $q$ to the outgoing angle

\[
q = \frac{2p \sin(\theta/2)}{\left[ 1 + \left( \frac{2E}{M_p} \right) \sin^2(\theta/2) \right]^{1/2}}
\]

where $p$ and $E$ are the momentum and energy of the incident electron in the lab frame
Interpreting Form Factors

• If proton is not pointlike cross section modified

\[
\frac{d\sigma}{d\Omega} \rightarrow \left( \frac{d\sigma}{d\Omega} \right)_{\text{pointlike}} |F(q^2)|^2
\]

• Finite size of scattering center introduces a phase difference between plane waves scattered from different points in space

From Thomson
Hoffstader and McAllister (1956)

Fig. 2. Arrangement of parts in experiments on electron scattering from a gas target.

Fig. 5. Curve (a) shows the theoretical Mott curve for a spinless point proton. Curve (b) shows the theoretical curve for a point proton with the Dirac magnetic moment, curve (c) the theoretical curve for a point proton having the anomalous contribution in addition to the Dirac value of magnetic moment. The theoretical curves (b) and (c) are due to Rosenbluth. The experimental curve falls between curves (b) and (c). This deviation from the theoretical curves represents the effect of a form factor for the proton and indicates structure within the proton, or alternatively, a breakdown of the Coulomb law. The best fit indicates a size of $0.70 \times 10^{-13}$ cm.

$$< r^2 >^\frac{1}{2} = 0.74 \pm 0.24 \times 10^{-13} \text{ cm} \sim 0.7 \text{ fm}$$
• For elastic scattering, the angle uniquely determines the energy of the outgoing electron
  ▶ So angle is the only independent variable
• Can write down the most general form of the matrix element

\[ \mathcal{M} = \frac{2\pi\alpha}{q^2} J^\text{electron}_\mu(q) J^\mu\text{ proton}(q) \]

▶ Electron is a Dirac particle, so we know \( J^\text{electron}_\mu(q) = e\bar{\psi}(q)\gamma_\mu\psi \)
▶ For proton, write down the most general form allowed by Lorentz invariance and parity conservation

\[ J^\text{proton}_\mu = \bar{\psi}(p_f) \left[ F_1(q^2)\gamma_\mu + \frac{i q^\nu \sigma_{\mu\nu}\kappa}{2M} F_2(q^2) \right] \psi(p_i) \]

where \( p_i \) and \( p_f \) are the initial and final proton four-momenta, \( q = k_i - k_f = p_f - p_i \) is the four-momentum transfer and \( \kappa \) is the anomalous magnetic moment of the proton
• Using the above, can calculate the cross section:

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2 \left(\frac{1}{2}\theta\right)}{4E^2 \sin^4 \left(\frac{1}{2}\theta\right)} \frac{E'}{E} \left[ \left( F_1^2 + \frac{\kappa^2 Q^2}{4M^2} F_2^2 \right) + \frac{Q^2}{2M^2} (F_1 + \kappa F_2)^2 \tan^2 \left(\frac{1}{2}\theta\right) \right]
\]

• \(\kappa\) is the anomalous magnetic moment

• Understanding these two form factors tells us about the structure of the proton

• In practice, better to use linear combinations of \(F_1\) and \(F_2\) defined so no interference terms occur in cross section

\[
G_E \equiv F_1 + \frac{\kappa q^2}{4M^2} F_2 \\
G_M \equiv F_1 + \kappa F_2 \\
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \left(\frac{1}{2}\theta\right)} \frac{E'}{E} \left[ \frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \left(\frac{\theta}{2}\right) - 2\tau G_M^2 \sin^2 \left(\frac{\theta}{2}\right) \right]
\]

with \(\tau \equiv -\frac{q^2}{4M^2}\)
Elastic Scattering: $G_E$ and $G_M$

- Reminder from previous page:

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \left(\frac{\theta}{2}\right)} \frac{E'}{E} \left[ \frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \left(\frac{\theta}{2}\right) - 2\tau G_M^2 \sin^2 \left(\frac{\theta}{2}\right) \right]
\]

$\tau = -q^2/4M^2 > 0$ is a Lorentz invariant

- Clear physical interpretations of $G_E$ and $G_M$
  
  ▶ In limit $\tau \ll 1$:

\[
G_E(q^2) = \int e^{i\vec{v} \cdot \vec{r}} \rho(\vec{r}) d^3 \vec{r}
\]

\[
G_M = e^{i\vec{v} \cdot \vec{r}} \mu(\vec{r}) d^3 \vec{r}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \left(\frac{\theta}{2}\right)} \frac{E'}{E} \left[ G_E^2 \cos^2 \left(\frac{\theta}{2}\right) \right]
\]

▶ In limit $\tau \gg 1$:

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \left(\frac{\theta}{2}\right)} \frac{E'}{E} \left[ 1 + 2\tau \tan^2 \left(\frac{\theta}{2}\right) G_M^2(q^2) \right]
\]

- Can measure $G_E$ and $G_M$ using angular dependence of cross section at fixed $q^2$
Measuring $G_E$ and $G_M$

$G_M(q^2) = 2.79G_E(q^2)$

Electric and magnetic form facts have same distribution
Elastic Scattering Summary

- Elastic scattering of relativistic electrons from a point-like Dirac proton
  \[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta / 2} \frac{E_3}{E_1} \left( \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} \right) \]

  - Rutherford
  - Proton recoil
  - Electric/magnetic scattering
  - Magnetic term due to spin

- Elastic scattering of relativistic electrons from an extended proton
  \[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta / 2} \frac{E_3}{E_1} \left( \frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right) \]

  - Rosenbuth formula

- Electron-proton elastic scattering demonstrates proton is an extended object with charge radius of \( \sim 0.8 \text{ fm} \)
What is the proton made of?

- Is the proton a soft mush or does it have hard composite objects inside?
- Need a high energy probe to resolve distances well below proton size
- Elastic cross section falls rapidly with $q^2$
- Inelastic cross section where proton breaks dominates rate at large $q^2$
  - “Deep inelastic scattering”
- Study energy and angle of outgoing electron
  - For inelastic scattering these are independent variables (subject to kinematic bounds of energy and momentum conservation)
Need High Energy Lepton Probe

Stanford Linear Collider (SLAC)
- Two mile linear accelerator ($e^{-}$)
- Initial phase: energy = 20 GeV
- (Later, upgrade to 50 GeV)
- “End Station A” hall for fixed target experiments

- Study high momentum transfer
  - Need four-momentum transfer large enough to probe structure
  - Proton breaks apart
  - Deep Inelastic Scattering (DIS)
The SLAC-MIT DIS Experiment (1968)
Deep Inelastic Scattering: Kinematics

- $W$ is the invariant mass of the hadronic system
- In lab frame: $P = (M, 0)$
- In any frame, $k = k' + q$, $W = p + q$
- Invariants of the problem:

\[
Q^2 = -q^2 = - (k - k')^2 \\
= 2EE'(1 - \cos \theta) \quad [\text{in lab}]
\]

\[
P \cdot q = P \cdot (k - k') \\
= M(E - E') \quad [\text{in lab}]
\]

- Define $\nu \equiv E - E'$ (in lab frame) so $P \cdot q = m\nu$ and

\[
W^2 = (P + q)^2 = (P - Q)^2 = M^2 + 2P \cdot q - Q^2 = M^2 + 2M\nu - Q^2
\]

where $Q^2 = -q^2$

- Elastic scattering corresponds to $W^2 = P^2 = M^2$
  - $Q^2 = 2M\nu$ elastic scattering
- We can define 2 indep dimensionless parameters

\[
x \equiv \frac{Q^2}{2M\nu}; \quad (0 < x \leq 1)
\]

\[
y \equiv \frac{P \cdot q}{P \cdot k} = 1 - E'/E; \quad (0 < y \leq 1)
\]
Deep Inelastic Scattering: Observation

Deep inelastic scattering
proton breaks up
resulting in a
many particle final
state
DIS = large W

Elastic Scattering
proton is intact
W=M

Inelastic scattering
produce excited states of the proton e.g. \( \Delta^+(1232) \)
\[ W = M_\Delta \]
The Most General Form of the Interaction

- Express cross section
  \[ d\sigma = L_{\mu\nu}^e W^{\mu\nu} \]
  where \( W \) describes the proton current (allowing substructure)
- Most general Lorentz invariant form of \( W^{\mu\nu} \)
  - Constructed from \( g^{\mu\nu}, p^\mu \) and \( q^\mu \)
  - Symmetric under interchange of \( \mu \) and \( \nu \) (otherwise vanishes when contracted with \( L_{\mu\nu} \))
  \[ W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^\mu p^\nu + \frac{W_4}{M^2} q^\mu q^\nu + \frac{W_5}{M^2} (p^\mu q^\nu + p^\nu q^\mu) \]
- \( W_3 \) reserved for parity violating term (needed for \( \nu \) scattering)
- Not all 4 terms are independent. Using \( \partial_\mu J^\mu = 0 \) can show
  \[ W_5 = -\frac{p \cdot q}{q^2} W_2 \]
  \[ W_4 = \frac{p \cdot q}{q^2} W_2 + \frac{M^2}{q^2} W_1 \]
  \[ W^{\mu\nu} = W_1 (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) + W_2 \frac{1}{M^2} (p^\mu - \frac{p \cdot q}{q^2} q^\mu)(p^\nu - \frac{p \cdot q}{q^2} q^\nu) \]
• Using notation from previous page, we can express the $x$-section for DIS

$$\frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2} \frac{\cos^2(\frac{1}{2}\theta)}{\sin^4(\frac{1}{2}\theta)} \left[ W_2(q^2, W) + 2W_1(q^2, W) \tan^2(\frac{1}{2}\theta) \right]$$

• These are the same two terms as for the elastic scattering

• $W_1$ and $W_2$ care called the $structure functions$
  - Angular dependence here comes from expressing covariant form on last page in lab frame variables
  - Two structure functions that each depend on $Q^2$ and $W$
  - Alternatively, can parameterize wrt dimensionless variables:

$$x \equiv \frac{Q^2}{2M\nu}$$

$$y \equiv \frac{P \cdot q}{P \cdot k} = 1 - \frac{E'}{E}$$
Studying the Proton at Large Momentum Transfer

- SLAC-MIT group measured $d\sigma/dq^2d\nu$ at 2 angles: $6^\circ$ and $10^\circ$

- For low $W$ dominated by production of resonances

- Surprise: Above the resonance region, $\sigma$ did not fall with $Q^2$

- Like Rutherford scattering, this is evidence for hard structure within the proton
Evidence for Hard Substructure

\[ \frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E'^2} \frac{\cos^2(\frac{1}{2}\theta)}{\sin^4(\frac{1}{2}\theta)} \left[ W_2(q^2, W) + 2W_1(q^2, W) \tan^2(\frac{1}{2}\theta) \right] \]

- How should we parameterize this deviation from behaviour predicted for pointlike proton?
  - To determine \( W_1 \) and \( W_2 \) separately, would need to measure at 2 values of \( E' \) and of \( \theta \) that give the same \( q^2 \) and \( \nu \)
  - The first exp couldn’t do this: small angle where experiment ran, \( W_2 \) dominates so studied that

- Once \( W \) and \( Q^2 \) large enough, cross section does not fall with \( Q^2 \)
  - As with Rutherford, evidence for hard objects within the
One more change of variables:

\[ F_1(x, Q^2) \equiv MW_1(\nu, Q^2) \]
\[ F_2(x, Q^2) \equiv \nu W_2(\nu, Q^2) \]

- Reminder: \( x \equiv Q^2 / 2M\nu \)
- Study \( F_2 \) for various energies and angles
- When low \( Q^2 \) data excluded, \( F_2 \) appears to depend only on dimensionless variable \( x \) and not on \( Q^2 \)
- This phenomenon is called “scaling”
Supposed there are pointlike partons inside the nucleon
Work in an “infinite momentum” frame: ignore mass effects
Proton 4-momentum: \( P = (P, 0, 0, P) \)
Visualize stream of parallel partons each with 4-momentum \( xP \) where \( 0 < x < 1 \); neglect transverse motion of the partons
- \( x \) is the fraction of the proton’s momentum that the parton carries
If electron elastically scatters from a parton
\[
(xP + q)^2 = m^2 \approx 0
\]
\[
x^2P^2 + 2xP \cdot q + q^2 = 0
\]
Since \( P^2 = M^2 \), if \( x^2M^2 << q^2 \) then
\[
2xP \cdot q = -q^2 = Q^2
\]
\[
x = \frac{Q^2}{2P \cdot q} = \frac{q^2}{2M \nu}
\]
This \( x \) is the same \( x \) we defined before!
Deep inelastic scattering can be described as elastic scattering of the lepton with a parton with momentum \( xP \)
What does Scaling Tell Us? (II)

- Suppose the partons in the proton have a distribution of fractional momentum $f(x)$
  - $f(x)dx = \text{probability of finding a parton carrying a fraction of the proton's momentum between } x \text{ and } x = dx$

- We can write the inelastic $ep$ scattering cross section as an incoherent sum of elastic scatters off the partons inside the proton

$$\frac{d\sigma^{ep}}{dEd\Omega} = \sum_i \int_0^1 f(x) \frac{d\sigma^{ei}}{dEd\Omega}$$

where the sum over $i$ is a sum over partons

- The cross section only depends on $x = q^2/2M\nu$ because that is the combination that picks out the momentum fraction carried by the parton
What Have We Learned?

Scaling of the Structure Functions is evidence for the presence of pointlike partons with the proton!

Some comments:

• We are using an impulse approximation where the scattering occurs before the partons have a chance to redistribute themselves

• We implicitly assume that after the scattering, the partons that participate in the scattering turn into hadrons with probability=1

• This is a lowest order calculation. We will see later that to higher order in perturbation theory, QCD corrections will introduce slow scaling violations ($Q^2$ dependence).