Wu et al showed that polarized Co\(^{60}\) decays had angular distribution:

\[
I(\theta) = 1 + \alpha \left( \frac{\sigma \cdot p}{E} \right) = 1 + \alpha \frac{\nu}{c} \cos \theta
\]

with later experiments verifying that \(\alpha = -1\).

- Charged current weak interactions are left handed, aside from mass terms.
- Garwin et al confirmed using \(\mu\)-decay.
- Goldhaber et al showed that \(\nu\) is left handed (and \(\bar{\nu}\) is right handed).
Reminder: Four Fermi Theory with V-A

- OK as long as $q^2 << M_W$
- Matrix element

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} J_1^\mu J_2^\mu$$

where $J_1, J_2$ are the two currents and $G_F \sim 10^{-5}$ GeV$^{-2}$

NB: The $\sqrt{2}$ is convention and we missing in Tues lecture
- We know now that currents exchange a $W^\pm$
- Leptonic current:

$$J_{\ell \, \mu} = \bar{\psi}_{\nu e \ell} (1 - \gamma_5) \psi_{\ell}$$

where $\psi$ is a spinor

$$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

- Note: $\gamma^5$ is a pseudoscalar, so leptonic current is $V - A$
- If momentum transfer is large, replace 4-point interaction with $W$-propagator
- This is called a “charged current” interaction since a $W^\pm$ is exchanged. We’ll get to “neutral currents” in Lecture 19
An aside on numerical factors

• I was sloppy Tuesday on numerical factors. Let’s be more careful this time.

• Before 1956 matrix element was defined

\[ M_{fi} = G_F g_{\mu\nu} \left[ \overline{\psi}_3 \gamma^\mu \psi_1 \right] \left[ \overline{\psi}_4 \gamma^\nu \psi_2 \right] \]

• Modification to add V-A interactions

\[ M_{fi} = \frac{1}{\sqrt{2}} G_F g_{\mu\nu} \left[ \overline{\psi}_3 \gamma^\mu \left( 1 - \gamma^5 \right) \psi_1 \right] \left[ \overline{\psi}_4 \gamma^\nu \left( 1 - \gamma^5 \right) \psi_2 \right] \]

where \( 1/\sqrt{2} \) is to keep value of \( G_F \) the same

• Replacing 4-Fermi interaction with propagator

\[ M_{fi} = \left[ \frac{g_W}{\sqrt{2}} \overline{\psi}_3 \gamma^\mu \left( 1 - \gamma^5 \right) \psi_1 \right] \left( \frac{-g^{\mu\nu} + q^\mu q^\nu / M_W^2}{q^2 - M_W^2} \right) \left[ \frac{g_W}{\sqrt{2}} \overline{\psi}_4 \gamma^\nu \left( 1 - \gamma^5 \right) \psi_2 \right] \]

• In limit \( q^2 << M_W \)

\[ M_{fi} = \frac{g_W^2}{8M_W^2} g_{\mu\nu} \left[ \overline{\psi}_3 \gamma^\mu \psi_1 \right] \left[ \overline{\psi}_4 \gamma^\nu \psi_2 \right] \]

\[ \Rightarrow \frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2} \]
Where we finished last time: Tau Decay

- $m_\tau = 1.777$ GeV
- Several possible decays:
  \[
  \begin{align*}
  \tau^- & \to e^- \bar{\nu}_e \nu_\tau \\
  \tau^- & \to \mu^- \bar{\nu}_\mu \nu_\tau \\
  \tau^- & \to d\bar{u} \nu_\tau
  \end{align*}
  \]
  In last case, the $d\bar{u}$ turns into hadrons with 100% probability
- All diagrams look like $\mu$-decay
- If $G_F^\mu = G_F^e = G_F$, predict:
  \[
  \Gamma_{\tau^- \to e^-} = \Gamma_{\tau^- \to \mu^-} = (m_\tau/m_\mu)^2 \Gamma(\mu)
  \]
  (difference in available phase space)

- Using the measured $\tau$-lifetime and BR, check consistency of $G_F$
  \[
  \frac{G_F^\tau}{G_F^\mu} = 1.0023 \pm 0.0033 \\
  \frac{G_F^e}{G_F^\mu} = 1.000 \pm 0.004
  \]
  Lepton universality for $G_F$
- For quark decays, need a factor of 3 for color. Predict
  \[
  BR(\tau \to \text{hadrons}) = \frac{3}{3 + 1 + 1} = 60%
  \]
- Experimental result:
  \[
  BR(\tau \to \text{hadrons}) = (64.76 \pm 0.06)\%
  \]
  Difference from 60% understood (QCD corrections; as for $R$)
Pion Decay (I) \[ \pi^- (q) \rightarrow \mu^- (p) + \bar{\nu}_\mu (k) \]

\[ \tau_{\pi^+} = 2.6 \times 10^{-8} \text{ s} \]

- $u\bar{d}$ annihilation into virtual $W^+$
- Depends on $\pi^+$ wave function at origin
  - Need phenomenological parameter that characterizes unknown wave function
- Write matrix element
  \[ M = \frac{G_F}{\sqrt{2}} J^\mu_\pi \bar{u}(p) \gamma_\mu (1 - \gamma_5) v(k) \]

where $J^\mu_\pi = f(q^2)q^\mu$ since $q^\mu$ is the only available 4-vector
- But $q^2 = m^2_\pi$ so $J_\pi = f_\pi q^\mu$. $f_\pi$ has units of mass (matrix element must be dimensionless)

- After spinor calculation, result for decay width:
  \[ \Gamma = \frac{G_F^2 f^2_\pi m_\pi m_\mu^2}{8\pi} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \]

- This came from:
  \[ |M|^2 \sim \frac{G_F^2 m_\mu^2 (m^2_\pi - m^2_\mu)}{8\pi m^2_\pi} f^2_\pi \]
  \[ \text{Phase Space} \sim \frac{|p|}{8\pi m^2_\pi} \]

- We'll examine what this means on the next page
From previous page

\[
\Gamma = \frac{G_F^2}{8\pi} f_\pi^2 m_\pi m_\mu \left( 1 - \frac{m_\mu^2}{m_\pi^2} \right)^2
\]

Result for electron same with \( m_\mu \to m_e \)

Thus

\[
\frac{\Gamma_e}{\Gamma_\mu} = \frac{m_e^2}{m_\mu^2} \left( \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \right)^2
\]

Since \( m_e = 0.51 \) MeV, \( m_\mu = 105.65 \) MeV and \( m_{\pi^+} = 139.57 \) MeV

\[
\frac{\Gamma_e}{\Gamma_\mu} \sim 1.2 \times 10^{-4}
\]

This agrees with measurements

Physically, result comes from helicity suppression

Spin 0 pion, right-handed antineutrino forces \( \mu^- \) to be right-handed

But \( \mu^- \) wants to be left-handed

\( \text{rh component} \sim (v/c)^2 \sim m_\mu \)

The less relativistic the decay product is, the larger the decay rate
Charged Kaon Decays

- $K^\pm$ mass larger than $\pi^\pm$
- More options for decay

**Leptonic**

Same calculation as for $\pi^\pm$
Helicity suppression make decay rate to muons larger that to electrons

$$BR(K^- \rightarrow \mu^- \bar{\nu}_\mu) = (63.56 \pm 0.11) \times 10^{-2}$$
$$BR(K^- \rightarrow e^- \bar{\nu}_e) = (1.582 \pm 0.007) \times 10^{-5}$$

**Semi-Leptonic**

3-body decay: No helicity suppression

$$BR(K^- \rightarrow \pi^0 \mu^- \bar{\nu}_\mu) = (3.352 \pm 0.033)\%$$
$$BR(K^- \rightarrow \pi^0 e^- \bar{\nu}_e) = (5.07 \pm 0.04)\%$$

More phase space for decay to $e$

**Hadronic (several diagrams possible)**

$$BR(K^- \rightarrow \pi^- \pi^0) = (20.67 \pm 0.08)\%$$
$$BR(K^- \rightarrow \pi^- \pi^0 \pi^0) = (1.760 \pm 0.023)\%$$
$$BR(K^- \rightarrow \pi^- \pi^- \pi^+) = (5.583 \pm 0.024)\%$$
Some Observations

- Leptonic decays of μ and τ demonstrate that $G_F$ the same for all lepton species

- Leptonic decay of charged pion and kaon tell us nothing about $G_F$ since $f_\pi$ and $f_K$ (which depend on wf at origin) are unknown

- If we want to ask whether $G_F$ is the same for hadronic currents as leptonic ones, we need to look at semileptonic decays
  
  ▶ Analog of β-decay

- But, well have to make sure that we are not affected by strong interaction corrections!
At low $q^2$ we measure WI using hadrons and not quarks.

Need to worry about whether binding of quarks in hadron affects the coupling:

$$1 - \gamma^5 \rightarrow C_V - C_A \gamma^5$$

Experimentally, for the neutron:

$$C_V = 1.00 \pm 0.003; \quad C_A = 1.26 \pm 0.02$$

- Vector coupling unaffected: protected by charge conservation (CVC)
- Axial vector coupling modified (PCAC)

Experimental implication: for precision tests of hadronic weak interactions, study decays that can only occur through $C_V$ term.

- This means decays between states of the same parity
- Best option is “superallowed” $\beta$-decay with $0^+ \rightarrow 0^+$ transition
- In addition to no axial vector component, such transitions cannot occur via $\gamma$ decay
Is $G_F$ Really Universal?

- Muon decay rate is

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

in approximation where $m_e$ ignored

- Same formula holds for nuclear $\beta$-decay

- A good choice of decay: $O^{14} \rightarrow N^{14*} e^+\nu_e (0^+ \rightarrow 0^+)$

- Correcting for available phase space we find

$$G_\mu = 1.166 \times 10^{-5}$$
$$G_\beta = 1.136 \times 10^{-5}$$

Close but not the same!

- What's going on?
Extend Our Study to More Hadron Decays

- Compare the following:

<table>
<thead>
<tr>
<th>Decay</th>
<th>Quark Level Decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \rightarrow ne^+\nu_e$</td>
<td>$u \rightarrow de^+\nu_e$</td>
</tr>
<tr>
<td>$\pi^- \rightarrow \pi^0 e^-\bar{\nu}_e$</td>
<td>$d \rightarrow ue^-\bar{\nu}_e$</td>
</tr>
<tr>
<td>$K^- \rightarrow \pi^0 e^-\bar{\nu}_e$</td>
<td>$s \rightarrow ue^-\bar{\nu}_e$</td>
</tr>
<tr>
<td>$\mu^+ \rightarrow \bar{\nu}_\mu e^+\nu_e$</td>
<td></td>
</tr>
</tbody>
</table>

- After correcting for phase space factors, $G_F$ obtained from $p$ and $\pi^-$ agree with each other, but are slightly less than obtained from $\mu$.
- $G_F$ obtained from $K^-$ decay appears much smaller.
- Either $G_F$ is not universal, or something else is going on!
Suppose strong and weak eigenstates of quarks not the same

Weak coupling:

Here \( d' \) is an admixture of down-type quarks

Normalization of w.f. for quarks means if \( d' = \alpha d + \beta s \), then

\[ \sqrt{\alpha^2 + \beta^2} = 1 \]

Can force this normalization by writing \( \alpha \) and \( \beta \) in terms of an angle

\[ d' = d \cos \theta_C + s \sin \theta_c \]

or

\[
\begin{pmatrix}
    d' \\
    s'
\end{pmatrix} =
\begin{pmatrix}
    \cos \theta_c & \sin \theta_C \\
    -\sin \theta_c & \cos \theta_c
\end{pmatrix}
\begin{pmatrix}
    d \\
    s
\end{pmatrix}
\]
• Using

\[ d' = d \cos \theta_C + s \sin \theta_C \]

we predict

- \( p \) & \( \pi \) decay \( \propto G_F^2 \cos^2 \theta_C \)
- \( K \) decay \( \propto G_F^2 \sin^2 \theta_C \)
- \( \mu \) decay \( \propto G_F^2 \)

• Using experimental measurements, find

\[
\begin{align*}
\cos \theta_C &= 0.97420 \pm 0.00021 \\
\sin \theta_C &= 0.2243 \pm 0.0005
\end{align*}
\]

• However, in addition to the \( d' \) there is an orthogonal down-type combination

\[ s' = s \cos \theta_C - d \sin \theta_C \]

Does it interact weakly?
A New Quark (Discussion from the Early 1970’s)

- It’s odd to have one charge $2/3$ quark and two charge $-1/3$ quarks
- Suppose there is a heavy $4^{th}$ quark
- We could then have two families of quarks. In strong basis:

\[
\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}
\]

Call this new quark “charm”
- Then, the weak basis is

\[
\begin{pmatrix} u' \\ d' \end{pmatrix} = d \cos \theta_C + s \sin \theta_C, \quad \begin{pmatrix} c' \\ s' \end{pmatrix} = s \cos \theta_C - d \sin \theta_C
\]

- There is a good argument for this charm quark in addition to $G_F$ ...
The GIM Mechanism (I)

- Glashow, Iliopoulous, Maiani (GIM) proposed existence of this 4th quark (charm)
- Charm couples to the $s'$ in the same way $u$ couples to the $d'$
- Reason for introducing charm: to explain why flavor changing neutral currents (FCNC) are highly suppressed
- Two examples of FCNC suppression:
  1. $BR(K^0_L \rightarrow \mu^+\mu^-) = 6.84 \times 10^{-9}$
  2. $BR(K^+ \rightarrow \pi^+\nu\nu)/BR(K^+ \rightarrow \pi^0\mu\nu) < 10^{-7}$
- Why are these decay rates so small?
- It turns out that there is also a $Z$ that couples to $f\bar{f}$ pairs, but it does not change flavor (same as $\gamma$)
- If only vector boson was the $W^\pm$, would require two bosons to be exchanged
  - Need second order charged weak interactions, but even this would give a bigger rate than seen unless there is a cancellation
The GIM Mechanism (II)

- Consider the “box” diagram

- $\mathcal{M}$ term with $u$ quark $\propto \cos \theta_C \sin \theta_C$
- $\mathcal{M}$ term $c$ quark $\propto -\cos \theta_C \sin \theta_C$
- Same final state, so we add $\mathcal{M}$’s
- Terms cancel in limit where we ignore quark masses
This Cancellation is Not a Accident!

- Matrix relating strong basis to weak basis is unitary
  \[ d'_i = \sum_j U_{ij} d_j \]

- Therefore is we sum over down-type quark pairs
  \[ \sum_i \overline{d}'_i d'_i = \sim_{ijk} \overline{d}_j U_{ji}^\dagger U_{ik} d_k \]
  \[ = \sum_j \overline{d}_j d_j \]

- If an interaction is diagonal in the weak basis, it stays diagonal in the strong basis

- Independent of basis, there are no \( d \longleftrightarrow s \) transitions

No flavor changing neutral current weak interactions
(up to terms that depend on the quark masses)
Some Questions and Answers about the GIM Mechanism

- Why is mixing in the down sector?
  - This is convention.
  - Charged current interactions always involve an up-type and a down-type quark
  - Can always define basis to move all mixing into either up or down sector

- Why is there no Cabbibo angle in the lepton sector?
  - Actually, there is!
  - Before people observed neutrino oscillations, they thought $\nu$’s were massless.
  - If all $\nu$ were massless, or had same mass, then free to redefine flavor basis to remove the mixing
  - We now need to define mixing angles for neutrinos as well as quarks
More Than Two Generations

• Generalize to $N$ families of quark ($N = 3$ as far as we know)

• $U$ is a unitary $N \times N$ matrix and $d'_i$ is an $N$-column vector

$$d'_i = \sum_{j=1}^{N} Y_{ij} d_j$$

• How many independent parameters do we need to describe $U$?
  ▶ $N \times N$ matrix: $N^2$ elements
  ▶ But each quark has an unphysical phase: can remove $2N - 1$ phases (leaving one for the overall phase of $U$)
  ▶ So, $U$ has $N^2 - (2N - 1)$ independent elements

• However, an orthogonal $N \times N$ matrix has $\frac{1}{2}N(N - 1)$ real parameters
  ▶ So $U$ has $\frac{1}{2}N(N - 1)$ real parameters
  ▶ $N^2 - (2N - 1) - \frac{1}{2}N(N - 1)$ imaginary phases ($= \frac{1}{2}(N - 1)(N - 2)$)

• $N = 2$ 1 real parameter, 0 imaginary

• $N = 3$ 3 real parameters, 1 imaginary

• Three generations requires an imaginary phase: CP Violation inherent
The CKM Matrix

• Write hadronic current

\[ J^\mu = -\frac{g}{\sqrt{2}} (\bar{u} \, \bar{c} \, \bar{t}) \gamma_\mu \frac{(1 - \gamma_5)}{2} V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \]

• \( V_{CKM} \) gives mixing between strong (mass) and (charged) weak basis

• Often write as

\[ V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \]

• Wolfenstein parameterization:

\[ V_{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \]

Here \( \lambda \) is the \( \approx \sin \theta_C \).
Best Fit for CKM Matrix from PDG

- From previous page

\[ V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \]

- Impose Unitary and use all experimental measurements

\[
\begin{align*}
\lambda &= 0.22453 \pm 0.00044 \\
A &= 0.836 \pm 0.015 \\
\rho &= 0.122^{+0.018}_{-0.17} \\
\eta &= 0.355^{+0.12}_{-0.11}
\end{align*}
\]

- Result for the magnitudes of the elements is:

\[
\begin{pmatrix}
0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\
0.22438 \pm 0.00044 & 0.97359 \pm 0.00011 & 0.04214 \pm 0.00076 \\
0.00896 \pm 0.00024 & 0.04133 \pm 0.00074 & 0.999105 \pm 0.00032
\end{pmatrix}
\]

- We'll come back to this in Lecture 18