Lecture 10: The Structure of the Proton: Part II

Sept 25, 2018
Reminder: Deep Inelastic Scattering

- $W$ is the invariant mass of the hadronic system
- In lab frame: $P = (M, 0)$
- In any frame, $k = k' + q$, $W = p + q$
- Invariants of the problem:
  
  \[
  Q^2 = -q^2 = -(k - k')^2 = 2EE'(1 - \cos \theta) \quad \text{[in lab]}
  \]
  \[
  P \cdot q = P \cdot (k - k') = M(E - E') \quad \text{[in lab]}
  \]

- Define $\nu \equiv E - E'$ (in lab frame) so $P \cdot q = m\nu$ and
  
  \[
  W^2 = (P + q)^2 = (P - Q)^2 = M^2 + 2P \cdot q - Q^2 = M^2 + 2M\nu - Q^2
  \]

  where $Q^2 = -q^2$

- Elastic scattering corresponds to $W^2 = P^2 = M^2$
  
  \[
  Q^2 = 2M\nu \text{ elastic scattering}
  \]

- We can define 2 indep dimensionless parameters
  
  \[
  x \equiv Q^2/2M\nu; \quad (0 < x \leq 1)
  \]
  \[
  y \equiv \frac{P \cdot q}{P \cdot k} = 1 - E'/E; \quad (0 < y \leq 1)
  \]
Reminder: Structure Functions

- Using notation from previous page, we can express the $x$-section for DIS

$$
\frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2} \frac{\cos^2\left(\frac{1}{2}\theta\right)}{\sin^4\left(\frac{1}{2}\theta\right)} \left[ W_2(q^2, W) + 2W_1(q^2, W) \tan^2\left(\frac{1}{2}\theta\right) \right]
$$

- $W_1$ and $W_2$ are called the structure functions
  - Angular dependence here comes from expressing covariant form on last page in lab frame variables
  - Two structure functions that each depend on $Q^2$ and $W$
  - Alternatively, can parameterize wrt dimensionless variables:
    \[
    x \equiv \frac{Q^2}{2M\nu} \\
y \equiv \frac{P \cdot q}{P \cdot k} = 1 - \frac{E'}{E}
    \]
• Change variables

\[ F_1(x, Q^2) \equiv MW_1(\nu, Q^2) \]
\[ F_2(x, Q^2) \equiv \nu W_2(\nu, Q^2) \]

• Rewrite cross section in terms of \( x, y, Q^2 \)

\[
\frac{d^2\sigma^{ep}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ \left( 1 - y - \frac{M^2 y^2}{Q^2} \right) \frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2) \right]
\]

• In DIS limit, \( Q^2 >> M^2 y^2 \):

\[
\frac{d^2\sigma^{ep}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ (1 - y) \frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2) \right]
\]

• Can event-by-event determine \( x, y \) and \( Q^2 \) from lab frame variables

\[ Q^2 = 4EE' \sin^2 \frac{\theta}{2}, \quad x = \frac{Q^2}{2M(E - E')} = \frac{Q^2}{2M\nu} =, \quad y = 1 - \frac{E'}{E} \]
Reminder: The Parton Model

- Supposed there are pointlike partons inside the nucleon
- Work in an “infinite momentum” frame: ignore mass effects
- Proton 4-momentum: $P = (P, 0, 0, P)$
- Visualize stream of parallel partons each with 4-momentum $xP$ where $0 < x < 1$; neglect transverse motion of the partons
  - $x$ is the fraction of the proton’s momentum that the parton carries
- If electron elastically scatters from a parton

\[
(xP + q)^2 = m^2 \simeq 0
\]
\[
x^2 P^2 + 2xP \cdot q + q^2 = 0
\]

Since $P^2 = M^2$, if $x^2 M^2 \ll q^2$ then
\[
2xP \cdot q = -q^2 = Q^2
\]
\[
x = \frac{Q^2}{2P \cdot q} = \frac{q^2}{2M \nu}
\]

Deep inelastic scattering can be described as elastic scattering of the lepton with a parton with momentum $xP$
Electron Quark Scattering

- Quarks are Dirac particles, so can just calculate the scattering in QED
- We won’t do the calculation here (see Thomson p. 191). Answer is

\[
\frac{d\sigma^{eq}}{dQ^2} = \frac{4\pi e_i^2}{Q^4} \left[(1 - y) + \frac{y}{2}\right]
\]

- Looks pretty similar to the previous page

\[
\frac{d^2\sigma^{ep}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[(1 - y) \frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2)\right]
\]

- The \( F_1(x, Q^2) \) and \( F_2(x, Q^2) \) carry the information about the distribution of the quarks inside the proton
- Note: We say last time that \( F_1(x, Q^2) \) is due to the magnetic interactions while \( F_2(x, Q^2) \) is the electric interaction
  - If partons are Dirac particles, we expect a well defined relationship between these two terms
Convolution of PDF with scattering cross section

• Cross section is incoherent sum over elastic scattering with partons

\[
\frac{d\sigma^{eq}}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ (1 - y) + \frac{y}{2} \right]
\]

\[
\frac{d\sigma^{ep}}{dx dQ^2} = \int_0^1 dx \sum_i e_i^2 f_i(x, Q^2) \frac{4\pi\alpha^2}{Q^4} \left[ (1 - y) + \frac{y}{2} \right] \delta(x - \frac{Q^2}{2M_N})
\]

• Comparing to the previous expression for ep scattering

\[
\frac{d^2\sigma^{ep}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ (1 - y) \frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2) \right]
\]

We find

\[
F_2^{ep}(x, Q^2) = x \sum_i e_i^2 f_i(x, Q^2)
\]

\[
F_1^{ep}(x, Q^2) = \frac{1}{2} \sum_i e_i^2 f_i(x, Q^2)
\]

\[
\therefore F_2^{ep}(x, Q^2) = 2x F_1^{ep}(x, Q^2)
\]

• Last equation is called the Callan-Gross relation

• If partons had spin-0 rather than spin-\(\frac{1}{2}\), we would have found \(F_1 = 0\)
What does the data look like?

The partons act like spin-1/2 Dirac particles!
• $f_i(x)$ is the prob of finding a parton of species $i$ with mom fraction between $x$ and $x + dx$ in the proton.

• If the partons together carry all the momentum of the proton

$$\int dx \ x f(x) = \int dx \ x \sum_i f_i(x) = 1$$

where $\sum_i$ is a sum over all species of partons in the proton

• We call $f(x)$ the parton distribution function since it tells us the momentum distribution of the parton within the proton

• This is the first example of a “sum rule”
• It’s natural to associate the partons with quarks, but that’s not the whole story
• Because $ep$ scattering occurs through the electromagnetic interaction, it only occurs via scattering with charged partons.
• If the proton also contains neutral partons, the EM scattering won't “see” them
  ▶ For example: EM scattering blind to gluons
• Let’s assume that the $ep$ scattering occurs through the scattering of the $e$ off a quark or antiquark
  ▶ We saw that the SU(3) description of the proton consists of 2 $u$ and 1 $d$ quark.
  ▶ However we can in addition have any number of $q\bar{q}$ pairs without changing the proton’s quantum numbers
  ▶ The 3 quarks ($uud$) are called valence quarks. The additional $q\bar{q}$ pairs are called sea or ocean quarks.
    • Pair production of $q\bar{q}$ pairs within the proton
To get the right quark content for the proton:

\[
\begin{align*}
\int u(x) - \bar{u}(x) \, dx &= 2 \\
\int d(x) - \bar{d}(x) \, dx &= 1 \\
\int s(x) - \bar{s}(x) \, dx &= 0
\end{align*}
\]
If partons are quarks, what do we expect?

- Elastic scattering from proton has \( x = 1 \)
- If 3 quarks carry all the proton’s momentum each has \( x = 0.3 \)
- Interactions among quarks smears \( f(x) \)
- Radiation of gluons softens distribution and adds \( q\bar{q} \) pairs
  - Describe the 3 original quarks as “valence quarks”
  - \( q\bar{q} \) pairs as sea or ocean
- Some of proton’s momentum carried by gluons and not quarks or antiquarks
Using Isospin: Comparing the Proton and Neutron

- Ignore heavy quark content in the proton: consider only $u$, $d$, $s$
- Write the proton Structure Function

\[
\frac{F_{2}^{p}(x)}{x} = \sum_{i} f_{i}^{p}(x)e_{i}^{2} = \frac{4}{9}(u^{p}(x) + \bar{u}^{p}(x)) + \frac{1}{9}(d^{p}(x) + \bar{d}^{p}(x)) + \frac{1}{9}(s^{p}(x) + \bar{s}^{p}(x))
\]

- Similarly, for the neutron

\[
\frac{F_{2}^{n}(x)}{x} = \sum_{i} f_{i}^{n}(x)e_{i}^{2} = \frac{4}{9}(u^{n}(x) + \bar{u}^{n}(x)) + \frac{1}{9}(d^{n}(x) + \bar{d}^{n}(x)) + \frac{1}{9}(s^{n}(x) + \bar{s}^{n}(x))
\]

- But isospin invariance tells us that $u^{p}(x) = d^{n}(x)$ and $d^{p}(x) = u^{n}(x)$
- Write $F_{2}$ for the neutron in terms of the proton pdf’s (assuming same strange content for the proton and neutron)

\[
\frac{F_{2}^{n}(x)}{x} = \frac{4}{9}(d^{p}(x) + \bar{d}^{p}(x)) + \frac{1}{9}(u^{p}(x) + \bar{u}^{p}(x)) + \frac{1}{9}(s^{p}(x) + \bar{s}^{p}(x))
\]

- Assuming sea $q$ and $\bar{q}$ distributions are the same:

\[
u(x) - \bar{u}(x) = u_{v}(x), \quad d(x) - \bar{d}(x) = d_{v}(x), \quad s(x) - \bar{s}(x) = 0
\]

- Taking the difference in $F_{2}$ for protons and neutrons:

\[
\frac{1}{x}[F_{2}^{p}(x) - F_{2}^{n}(x)] = \frac{1}{3}[u_{v}(x) - d_{v}(x)]
\]

which gives us a feel for the valence quark distribution
What the data tells us

From Halzen and Martin

- Looks the way we expect from the cartoon on page 27
- Next question: How to measure the partons’ charge
  - To do this, must compare $e$ and $\nu$ scattering!

Fig. 9.8 The difference $F_2^p - F_2^n$ as a function of $x$, as measured in deep inelastic scattering. Data are from the Stanford Linear Accelerator.
Neutrino-(anti)quark Charged Current Scattering (I)

- Start with $\nu_\mu$ or $\bar{\nu}_\mu$ beam
  - Distribution of ingoing $\nu$ 4-momenta determined from beam design
  - Outgoing $\mu^\pm$ momentum measured in spectrometer

- Exchange via $W^\pm$ (“charged current interaction”)
  - $\nu$ scatter against $\bar{d}$ and $u$
  - $\bar{\nu}$ scatter against $u$ and $\bar{d}$

We’ll talk about neutral currents in a few weeks
Not useful for structure function measurements
(Can’t measure outgoing lepton 4-momentum)
Neutrino-(anti)quark Scattering (II)

- Neutrinos left handed, anti-neutrinos right handed
- Left handed $W^\pm$ couples to left-handed quarks and right-handed anti-quarks

\[
\frac{d\sigma_{\nu q}}{d\cos\theta} \propto \text{constant} \quad \frac{d\sigma_{\bar{\nu}q}}{d\cos\theta} \propto (1 + \cos\theta^*)^2
\]

where $\theta^*$ is scattering angle in $\nu q$ center of mass

- $\nu q$ and $\bar{\nu}q$ scattering allowed for all angles, but $\bar{\nu}q$ and $\nu\bar{q}$ vanish in backward direction

- We’ll see later that this left-handed coupling is also reason that $\pi$ and $K$ preferentially decay to $\mu$ and not $e$
  
  - $\mu$ needs to be right-handed since $\pi, K$ have spin 0
  - rh component of spinor $\propto (v/c) \propto m_\mu$ in matrix element;
    decay rate $\Gamma \propto m_\mu^2$

This is why accelerators produce predominantly $\nu\mu, \bar{\nu}\mu$
Neutrino-(anti)quark Scattering (III)

- The charged current cross sections are $\nu_\mu$:

$$\frac{d\sigma(\nu_\mu \, d \rightarrow \mu^- \, u)}{d\Omega} = \frac{G_F^2 \cdot s}{4\pi^2}$$

$$\frac{d\sigma(\nu_\mu \, u \rightarrow \mu^+ \, d)}{d\Omega} = \frac{G_F^2 \cdot s \cdot (1 + \cos \theta)^2}{4\pi^2}$$

$$\frac{d\sigma(\nu_\mu \, \bar{u} \rightarrow \mu^- \, \bar{d})}{d\Omega} = \frac{G_F^2 \cdot s \cdot (1 + \cos \theta)^2}{4\pi^2}$$

$$\frac{d\sigma(\nu_\mu \, \bar{d} \rightarrow \mu^+ \, \bar{u})}{d\Omega} = \frac{G_F^2 \cdot s}{4\pi^2}$$

- You will prove on homework #5 that

$$1 - y = \frac{p \cdot k'}{p \cdot k} = \frac{1}{2} \cdot (1 + \cos \theta^*)$$

which allows us to rewrite the above expressions in terms of the relativistically invariant variable $y$

- Since $\int \frac{(1+\cos \theta)^2}{4} d \cos \theta = 1/3$,

$$\sigma^{\nu d} : \sigma^{\nu \bar{u}} : \sigma^{\bar{\nu} u} : \sigma^{\bar{\nu} \bar{d}} = 1 : \frac{1}{3} : \frac{1}{3} : 1$$
What is the Advantage of $\nu$ Scattering?

- The quarks and antiquarks have different angular dependence, so we can extract their pdf’s separately by looking at cross sections as a function of angle
  - Angular dependence can be expressed in terms of dimensionless variable $y$
  - Parity violation means we have a third structure function $F_3$ that I won’t talk about today
- Weak “charge” of the $u$ and $d$ is the same, so factors of $4/9$ and $1/9$ are not present
- Using previous expressions and integrating over angle:

$$\frac{d\sigma(\nu p)}{dx} = \frac{G_F^2 x_s}{\pi} \left[ d(x) + \frac{1}{3} \bar{u}(x) \right]$$

$$\frac{d\sigma(\nu n)}{dx} = \frac{G_F^2 x_s}{\pi} \left[ d^n(x) + \frac{1}{3} \bar{u}^n(x) \right]$$

$$= \frac{G_F^2 x_s}{\pi} \left[ u(x) + \frac{1}{3} \bar{d}(x) \right]$$

where we have written everything in terms of the proton PDFs
- If we believe the partons in the proton and neutron are quarks, we can relate the structure functions measured in $\nu N$ and $eN$
An Aside: How do we know the incoming neutrino energy?

- Primary proton beam incident on target produces secondary $\pi$ and $K^-$
- Use magnets and shielding to select range of momenta of secondaries
- Long decay region to allow the $\pi \rightarrow \mu \nu$ and $K \rightarrow \mu \nu$ decays
- Two body decay gives correlation between decay angle and neutrino momentum
Comparing $eN$ and $\nu N \nu N$ Scattering (I)

- Now, let’s take an isoscalar target $N$ (equal number of protons and neutrons)
- In analogy with electron scattering

$$\frac{F_2^{\nu N}}{x} = u(x) + d(x) + \bar{u}(x) + \bar{d}(x)$$

- If we go back to our electron scattering and also require an isoscalar target

$$\frac{F_2^{e N}}{x} = \frac{5}{18} \left( u(x) + d(x) + \bar{u}(x) + \bar{d}(x) \right)$$

- So, if the partons have the charges we expect from the quark model

$$F_2^{e N}(x) = \frac{5}{18} F_2^{\nu N}(x)$$
Comparing $eN$ and $\nu N$ Scattering (II)

- The partons we “see” in $eN$ scattering are the same as the ones we “see” in $\nu N$ scattering.

- This confirms our assignment of the quark charges:

  The Quarks Have Fractional Charge!
As we previously did for electron scattering, we can look at an isoscalar target $N$.

Starting with the cross sections for $\nu q$ scattering we can go through the same convolution with the PDFs that we did for the $eN$ case.

The result is

$$\sigma^{\nu N} = \frac{G_F 2ME}{2\pi} \left[ Q + \frac{1}{3} \overline{Q} \right]$$

$$\sigma^{\bar{n}uN} = \frac{G_F 2ME}{2\pi} \left[ \overline{Q} + \frac{1}{3} Q \right]$$

where

$$Q \equiv \int x[u(x) + d(x)]$$

$$\overline{Q} \equiv \int x[\bar{u}(x) + \bar{d}(x)]$$

and we have ignored the small strange component in the nucleon.

Thus

$$R_{\nu/\bar{\nu}} \equiv \frac{\sigma^{\bar{n}uN}}{\sigma^{\nu N}} = \frac{\overline{Q} + Q/3}{Q + \overline{Q}/3} = \frac{1 + 3\overline{Q}/Q}{3 + Q/\overline{Q}}$$
• Experimentally $R_{\nu/\bar{\nu}} = 0.45 \rightarrow \overline{Q}/Q = 0.5$

There are antiquarks within the proton!
How Much Momentum do the $q$ and $\bar{q}$ Carry?

- Momentum fraction that the $q$ and $\bar{q}$ together carry is

$$\int xF_2^\nu N(x)dx = \frac{18}{5} \int xF_2^{eN}(x)dx$$

- At $q^2 \sim 10$ GeV$^2$ that this fraction $\sim 0.5$

  Only half the momentum of the proton is carried by quarks and antiquarks

- What’s Left? The gluon!
• Charged lepton probes study charged partons
• Neutrinos study all partons with weak charge
  \[ \int x F_2^{\nu N}(x) dx = \frac{18}{5} \int x F_2^{e N}(x) dx \] tells us that all the weakly interacting partons are charged
• To study the gluon directly, will need a strong probe
  ▶ No pointlike strong probes
  ▶ Will need to convolute two pdf’s

▶ More on this when we talk about hadron colliders in a few weeks
• Can also indirectly study gluon by seeing how it affects the quarks
Scaling Violations in DIS

- QCD corrections to DIS come from incorporating gluon brem from the $q$ and $\bar{q}$ and pair production $g \rightarrow q\bar{q}$
- The ability to resolve these QCD corrections are $q^2$ dependent
- Expected result:
  - At high $x$ the quark pdf’s decrease
  - At low $x$ the quark and antiquark pdf’s increase
- Complete treatment in QCD via coupled set of differential equations, the Alterelli-Parisi evolution equations
DIS in the Modern Era: The HERA collider

- $ep$ collider located at DESY lab in Hamburg
- 27.5 GeV ($e$) x 920 GeV ($p$)
- Two general purpose detectors (H1 and Zeus)
What $Q^2$ and $x$ are relevant?
Our best fits of PDFs at present

- Fit experimental data to theoretically motivated parameterizations
- Combine data from many experiments, using Alterelli-Parisi to account for differences in $Q^2$ (correct to common value)
- Analysis of uncertainties to provide a systematic uncertainty band
Modern $F_2(x, Q^2)$ Measurements