Physics 137B (Professor Shapiro) Spring 2010

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Homework 9 Solutions

1. From equation 12.94 of the text, the energy levels of the hydrogen atom, taking into account fine structure and a weak-field Zeeman splitting, are $|n, l, j, m_j>$ with energy:

$$E = -\frac{13.6eV}{n^2}[1 + \frac{\alpha^2}{n^2}(\frac{n}{j+1/2} - \frac{3}{4})] + \mu_B B g_J m_j$$

where $g_J = 1 + \dfrac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}$. So in this case the eight n=2 states get split into (labeling the states by $|n, l, j, m_j>$):

$|2,0,1/2,1/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{5\alpha^2}{16}] + \mu_B B$

$|2,0,1/2,-1/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{5\alpha^2}{16}] - \mu_B B$

$|2,1,1/2,1/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{5\alpha^2}{16}] + \mu_B B/3$

$|2,1,1/2,-1/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{5\alpha^2}{16}] - \mu_B B/3$

$|2,1,3/2,3/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{\alpha^2}{16}] + 2\mu_B B$

$|2,1,3/2,1/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{\alpha^2}{16}] + 2\mu_B B/3$

$|2,1,3/2,-1/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{\alpha^2}{16}] - 2\mu_B B/3$

$|2,1,3/2,-3/2>$ $\quad E = -\frac{13.6eV}{4}[1 + \frac{\alpha^2}{16}] - 2\mu_B B$

2. In this case the magnetic field splitting dominates and we can ignore the fine structure. Therefore $|n, l, m_l, m_s>$ are the eigenstates with energy $E = -\frac{13.6eV}{n^2} + \mu_B B(m_l + 2m_s)$ (see equation 12.86 of the text). So labeling the states by $|n, l, m_l, m_s>$, we have:

$\|2,0,0,1/2>$	$E = -\frac{13.6eV}{4} + \mu_B B$
$\|2,0,0,-1/2>$	$E = -\frac{13.6eV}{4} - \mu_B B$
$\|2,1,1,1/2>$	$E = -\frac{13.6eV}{4} + 2\mu_B B$
$\|2,1,1,-1/2>$	$E = -\frac{13.6eV}{4}$
$\|2,1,0,1/2>$	$E = -\frac{13.6eV}{4} + \mu_B B$
$\|2,1,0,-1/2>$	$E = -\frac{13.6eV}{4} - \mu_B B$
$\|2,1,-1,1/2>$	$E = -\frac{13.6eV}{4}$
$\|2,1,-1,-1/2>$	$E = -\frac{13.6eV}{4} - 2\mu_B B$

3. In the scattering problem, the wavefunction satisfies Schrodinger's equation for a free particle, $(\Delta^2 + k^2)|\psi>= 0$ as $r \to \infty$. The incident wave is a plane wave and the scattered wave is a spherical wave multiplied by an arbitrary function of θ, $f(\theta)$.

In one dimension Schrodinger's equation in terms of $r = |x|$ becomes (for a spherical wave that depends only on r):

$$(\frac{d^2}{dr^2} + k^2)\psi_{sph}(r) = 0$$

This has solution $\psi_{sph}(r) = e^{ikr}$ and so the wavefunction for the scattering process is: $\psi(r, \theta := \text{sign}(x)) = A\{e^{ikx} + f(\theta)e^{ikr}\}$.

In two dimensions Schrodinger's equation in terms of $r = \sqrt{x^2 + y^2}$ becomes (for a spherical wave that depends only on r):

$$(\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}) + k^2)\psi_{sph}(r) = 0$$

Try a has solution of the form $\psi_{sph}(r) = \dfrac{e^{ikr}}{r^n}$:

$$\begin{aligned}
(\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}) + k^2)(\frac{e^{ikr}}{r^n}) &= \frac{1}{r}\frac{d}{dr}(-n\frac{e^{ikr}}{r^n} + ik\frac{e^{ikr}}{r^{n-1}}) + k^2\frac{e^{ikr}}{r^n} \\
&= (n^2\frac{e^{ikr}}{r^{n+2}} - ikn\frac{e^{ikr}}{r^{n+1}} - (n-1)ik\frac{e^{ikr}}{r^{n+1}} - k^2\frac{eikr}{r^n}) + k^2\frac{e^{ikr}}{r^n} \\
&= n^2\frac{e^{ikr}}{r^{n+2}} - (2n-1)ik\frac{e^{ikr}}{r^{n+1}} \\
&\approx -(2n-1)ik\frac{e^{ikr}}{r^{n+1}} \qquad \text{for large } r >> 1/k
\end{aligned}$$

which vanishes if $n = 1/2$. So the analogous wave function in 2 dimensions is $\psi(r,\theta) = A\{e^{ikx} + f(\theta)\dfrac{e^{ikr}}{r^{1/2}}\}$.

4. (a) The number of particles scattered per unit area per unit time is:

$$\tilde{N} = n\mathcal{N} = n\Phi\frac{d\sigma}{d\Omega}d\Omega$$

So the fraction of particles scattered into an angle $d\theta$ is:

$$\begin{aligned}
\frac{\tilde{N}}{\Phi} &= n\frac{d\sigma}{d\Omega}d\Omega \\
&= n(\frac{qQ}{4E})^2\frac{1}{\sin^4(\theta/2)}(2\pi|d(\cos\theta)|) \\
&= n(\frac{qQ}{4E})^2\frac{\sin\theta}{\sin^4(\theta/2)}(2\pi d\theta)
\end{aligned}$$

(b) $n =$ density $\times$ thickness $\times$ $N_0/A = 5.9 \times 10^{22}$ atoms$/m^2$. Substituting this value into the above equation and using $Q = 79e$, $q = 2e$, $\theta = \pi/2$ and $E = 5MeV$ we get that the fraction of atoms scattered into the cone is $1.9 \times 10^{-4} d\theta$

5. Starting with the equation:

$$R_l \rightarrow \frac{1}{kr}[B_l\sin(kr - l\pi/2) - C_l\cos(kr - l\pi/2)]$$

Then if we define $A_l := (B_l^2 + C_l^2)^{1/2}$ and $\delta_l := -\tan^{-1}[C_l/B_l]$ then by basic trigonometry, $B_l = A_l \cos(\delta_l)$ and $C_l = -A_l \sin(\delta_l)$ and so

$$\begin{aligned} B_l \sin(kr - l\pi/2) - C_l \cos(kr - l\pi/2) &= A_l(\cos(\delta_l)\sin(kr - l\pi/2) + \sin(\delta_l)\cos(kr - l\pi/2)) \\ &= A_l(\sin(k_r) - l\pi/2 + \delta_l) \end{aligned}$$

which gives equation 13.40 of the text.

6. By substituting equations 13.34, 13.35, 13.46 of the text into 13.44, we get that

$$\sum_{l=0}^{\infty} R_l(k,r)P_l(\cos\theta) \quad \to \quad \sum_{l=0}^{\infty}\{(2l+1)i^l(kr)^{-1}\sin(kr - l\pi/2) + r^{-1}\exp ikr f_l(k)\}P_l(\cos\theta)$$

Multiplying both sides of the above by $P_{l'}(\cos\theta)\sin\theta$ and integrating over $0 < \theta < \pi$, and using the orthogonality relation $\int_0^\pi P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$ we get that

$$\begin{aligned} R_l(k,r) &\to (2l+1)i^l(kr)^{-1}\sin(kr - l\pi/2) + r^{-1}\exp ikr f_l(k) \\ &\to \{(2l+1)i^l(kr)^{-1} + r^{-1}\exp(il\pi/2)f_l(k)\}\sin(kr - l\pi/2) \\ &\qquad + ir^{-1}\exp(il\pi/2)f_l(k)\cos(kr - l\pi/2) \end{aligned}$$

So, by comparing with 13.40 we have:

$$\begin{aligned} B_l(k) &= (2l+1)i^l + k\exp(il\pi/2)f_l(k) \\ C_l(k) &= -ik\exp(il\pi/2)f_l(k) \end{aligned}$$

and therefore:

$$\begin{aligned} A_l := (B_l^2 + C_l^2)^{1/2} = B_l(k) &= (2l+1)i^l + k\exp(il\pi/2)f_l(k) \\ C_l(k) &= -ik\exp(il\pi/2)f_l(k) \end{aligned}$$

From here it is a trivial but tedious algebraic exercise to obtain 13.48.