Physics 137B (Professor Shapiro) Spring 2010

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Homework 5 Solutions

1. (a) By the variational method $E(c^2) := \dfrac{<\phi_0|H|\phi_0>}{<\phi_0|\phi_0>}$ is an upper bound for the energy of the ground state.

$$
\begin{aligned}
<\phi_0|\phi_0> &= \int_{-\infty}^{\infty} |\phi_0|^2 dx \\
&= \int_{-c}^{c} (c^2 - x^2)^4 dx \\
&= \int_{-c}^{c} (c^8 - 4c^6x^2 + 6c^4x^4 - 4c^2x^6 + x^8) dx \\
&= (c^8x - 4c^6x^3/3 + 6c^4x^5/5 - 4c^2x^7/7 + x^9/9)|_{x=-c}^{x=c} \\
&= 2(c^9 - 4c^9/3 + 6c^9/5 - 4c^9/7 + c^9/9) \\
&= \frac{256c^9}{315}
\end{aligned}
$$

$$
\begin{aligned}
<\phi_0|\frac{1}{2}m\omega^2x^2|\phi_0> &= \frac{1}{2}m\omega^2 \int_{-\infty}^{\infty} \phi_0^* x^2 \phi_0 dx \\
&= \frac{1}{2}m\omega^2 \int_{-c}^{c} (c^2 - x^2)^4 x^2 dx \\
&= \frac{1}{2}m\omega^2 \int_{-c}^{c} (c^8x^2 - 4c^6x^4 + 6c^4x^6 - 4c^2x^8 + x^{10}) dx \\
&= \frac{1}{2}m\omega^2 (c^8x^3/3 - 4c^6x^5/5 + 6c^4x^7/7 - 4c^2x^9/9 + x^{11}/11)|_{x=-c}^{x=c} \\
&= \frac{1}{2}m\omega^2 2(c^{11}/3 - 4c^{11}/5 + 6c^{11}/7 - 4c^{11}/9 + c^{11}/11) \\
&= m\omega^2 \frac{128c^{11}}{3465}
\end{aligned}
$$

$$
\begin{aligned}
< \phi_0|\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}|\phi_0 > &= \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty}\phi_0^*\frac{d^2}{dx^2}\phi_0 dx \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}|\frac{d\phi_0}{dx}|^2 dx \qquad \text{(integrating by parts)} \\
&= \frac{\hbar^2}{2m}\int_{-c}^{c}[-4x(c^2-x^2)]^2 dx \\
&= \frac{\hbar^2}{2m}\int_{-c}^{c}[16x^2(c^4-2c^2x^2+x^4)]dx \\
&= \frac{\hbar^2}{2m}\int_{-c}^{c}(16c^4x^2-32c^2x^4+16x^6)dx \\
&= \frac{\hbar^2}{2m}(16c^4x^3/3-32c^2x^5/5+16x^7/7)|_{x=-c}^{x=c} \\
&= \frac{\hbar^2}{2m}2(16c^7/3-32c^7/5+16c^7/7) \\
&= \frac{\hbar^2}{m}\frac{128c^7}{105}
\end{aligned}
$$

So we have that:

$$
\begin{aligned}
E(c^2) &:= \frac{< \phi_0|H|\phi_0 >}{< \phi_0|\phi_0 >} \\
&= \frac{\frac{\hbar^2}{m}\frac{128c^7}{105}+m\omega^2\frac{128c^{11}}{3465}}{\frac{256c^9}{315}} \\
&= \frac{\hbar^2}{m}\frac{3}{2c^2}+m\omega^2\frac{c^2}{22}
\end{aligned}
$$

This is an upper bound for the ground state energy for every value of c^2. To find the lowest upper bound we need to find the minimum of this function.

$$
\begin{aligned}
\frac{dE(c^2)}{d(c^2)} &= -\frac{\hbar^2}{m}\frac{3}{2c^4}+\frac{m\omega^2}{22}=0 \\
\frac{\hbar^2}{m}\frac{3}{2c^4} &= \frac{m\omega^2}{22} \\
c^4 &= \frac{\hbar^2}{m}\frac{33}{m\omega^2} \\
c^2 &= \frac{\hbar}{m\omega}\sqrt{33}
\end{aligned}
$$

Then the minimum value of $E(c^2)$ is:

$$\begin{aligned} E_{\min} = E(\frac{\hbar}{m\omega}\sqrt{33}) &= \hbar\omega(\frac{3}{2\sqrt{33}} + \frac{\sqrt{33}}{22}) \\ &= \hbar\omega\frac{\sqrt{33}}{11} \\ &\approx 0.522\hbar\omega \end{aligned}$$

Therefore the ground state energy must be less than $0.522\hbar\omega$.

(b) The true ground state of the harmonic oscillator is $\psi_0(x) = (\frac{\alpha}{\sqrt{\pi}})^{1/2}e^{-\alpha^2x^2/2}$ which is an even function of x. But $\phi_1(x) = x\phi_0(x)$ is an odd function in x. Thus $< \phi_1|\psi_0 >= \int_{-\infty}^{\infty}\phi_1(x)\psi_0(x)dx$ is the integral of an odd function and thus is zero. So $\psi_1(x)$ is a suitable trial function for the first excited state. Then we have

$$\begin{aligned} < \phi_1|\phi_1 > &= \int_{-\infty}^{\infty}|\phi_1|^2dx \\ &= \int_{-c}^{c}x^2(c^2 - x^2)^4dx \\ &= \int_{-c}^{c}(c^8x^2 - 4c^6x^4 + 6c^4x^6 - 4c^2x^8 + x^10)dx \\ &= (c^8x^3/3 - 4c^6x^5/5 + 6c^4x^7/7 - 4c^2x^9/9 + x^{11}/11)|_{x=-c}^{x=c} \\ &= 2(c^{11}/3 - 4c^{11}/5 + 6c^{11}/7 - 4c^{11}/9 + c^{11}/11) \\ &= \frac{256c^{11}}{3465} \end{aligned}$$

$$\begin{aligned} < \phi_1|\frac{1}{2}m\omega^2x^2|\phi_1 > &= \frac{1}{2}m\omega^2\int_{-\infty}^{\infty}\phi_1^*x^2\phi_1dx \\ &= \frac{1}{2}m\omega^2\int_{-c}^{c}(c^2 - x^2)^4x^4dx \\ &= \frac{1}{2}m\omega^2\int_{-c}^{c}(c^8x^4 - 4c^6x^6 + 6c^4x^8 - 4c^2x^{10} + x^{12})dx \\ &= \frac{1}{2}m\omega^2(c^8x^5/5 - 4c^6x^7/7 + 6c^4x^9/9 - 4c^2x^{11}/11 + x^{13}/13)|_{x=-c}^{x=c} \\ &= \frac{1}{2}m\omega^2 2(c^{13}/5 - 4c^{13}/7 + 6c^{13}/9 - 4c^{13}/11 + c^{13}/13) \\ &= m\omega^2\frac{128c^{13}}{15015} \end{aligned}$$

$$
\begin{aligned}
< \phi_1 | \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} | \phi_1 > &= \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} \phi_1^* \frac{d^2}{dx^2}\phi_1 dx \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} |\frac{d\phi_1}{dx}|^2 dx \qquad \text{(integrating by parts)} \\
&= \frac{\hbar^2}{2m}\int_{-c}^{c} (5x^4 - 6c^2x^2 + c^4)^2 dx \\
&= \frac{\hbar^2}{2m}\int_{-c}^{c} (25x^8 - 60c^2x^6 + 46c^4x^4 - 12c^6x^2 + c^8) dx \\
&= \frac{\hbar^2}{2m}(25x^9/9 - 60c^2x^7/7 + 46c^4x^5/5 - 12c^6x^3/3 + c^8x)|_{x=-c}^{x=c} \\
&= \frac{\hbar^2}{2m}2(25c^9/9 - 60c^9/7 + 46c^9/5 - 12c^9/3 + c^9) \\
&= \frac{\hbar^2}{m}\frac{128c^9}{315}
\end{aligned}
$$

So we have that:

$$
\begin{aligned}
\widetilde{E}(c^2) &:= \frac{< \phi_1 | H | \phi_1 >}{< \phi_1 | \phi_1 >} \\
&= \frac{\frac{\hbar^2}{m}\frac{128c^9}{315} + m\omega^2\frac{128c^{13}}{15015}}{\frac{256c^{11}}{3465}} \\
&= \frac{\hbar^2}{m}\frac{11}{2c^2} + m\omega^2\frac{3c^2}{26}
\end{aligned}
$$

This is an upper bound for the first excited state energy for every value of c^2. To find the lowest upper bound we need to find the minimum of this function.

$$
\begin{aligned}
\frac{d\widetilde{E}(c^2)}{d(c^2)} &= -\frac{\hbar^2}{m}\frac{11}{2c^4} + \frac{3m\omega^2}{26} = 0 \\
\frac{\hbar^2}{m}\frac{11}{2c^4} &= \frac{3m\omega^2}{26} \\
c^4 &= \frac{\hbar^2}{m}\frac{143}{3m\omega^2} \\
c^2 &= \frac{\hbar}{m\omega}\sqrt{\frac{143}{3}}
\end{aligned}
$$

Then the minimum value of $\widetilde{E}(c^2)$ is:

$$\begin{aligned}
\widetilde{E}_{\min} = \widetilde{E}(\frac{\hbar}{m\omega}\sqrt{\frac{143}{3}}) &= \hbar\omega(\frac{11\sqrt{3}}{2\sqrt{143}} + \frac{3\sqrt{143}}{26\sqrt{3}}) \\
&= \hbar\omega\frac{\sqrt{3 \times 143}}{13} \\
&\approx 1.593\hbar\omega
\end{aligned}$$

Therefore the first excited state energy must be less than $1.593\hbar\omega$.

2. The infinite square well has energy eigenvalues $E_n = \alpha n^2/m$ corresponding to the one-particle energy eigenstate $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$, where n is a positive integer and $\alpha := \frac{\hbar^2\pi^2}{2L^2}$

(a) In this problem there are two identical neutrons (call these particles 1 and 2) and one proton (call this particle 3). The ground state of the combined system will correspond to all three particles in the one-particle square well n=1 level, giving total energy $E = \alpha(2/m_n + 1/m_p)$ (we will see that this state is allowed by considering the symmetrization properties in what follows). There is only one proton, so it does not need to satisfy any particular symmetrization. The proton can have spin up or down (2 possible states). The neutrons on the other hand must have an antisymmetric wavefunction under interchange of the two neutrons. If they are in the same spatial state (the ground state) of the infinite square well, their spatial wavefunction can only be symmetric. The neutrons must then have spins in the (antisymmetric) singlet state in order to make their total wavefunction antisymmetric (1 possible state). Thus the degeneracy of the ground state is 2, with the ground state space having a basis:

$$|\psi_1>_1 |\psi_1>_2 |\psi_1>_3 \frac{1}{\sqrt{2}}(|\uparrow>_1 |\downarrow>_2 - |\downarrow>_1 |\uparrow>_2)|\uparrow>_3$$
$$|\psi_1>_1 |\psi_1>_2 |\psi_1>_3 \frac{1}{\sqrt{2}}(|\uparrow>_1 |\downarrow>_2 - |\downarrow>_1 |\uparrow>_2)|\downarrow>_3$$

Each of these states is clearly antisymmetric under the interchange of the two neutrons (particles 1 and 2).

The first excited state of the combined system will have two particles in the square well n=1 level and one particle in the n=2 level. The neutron has a larger mass than the proton so it will be of lower energy to put one of the neutrons in the n=2 level (rather than the proton).

So the energy of the first excited state will be $E = \alpha(5/m_n + 1/m_p)$. Therefore, one of the neutrons is in the n=2 level with the other in the n=1 level and we can form either a symmetric or antisymmetric spatial state for the neutrons and combine it with an antisymmetric or symmetric neutron spin state respectively. The proton, meanwhile, is in the n=1 level and can have spin either up or down. So we have the following states:

$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 + |\psi_2>_1|\psi_1>_2)|\psi_1>_3 \frac{1}{\sqrt{2}}(|\uparrow>_1|\downarrow>_2 - |\downarrow>_1|\uparrow>_2)|\uparrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 + |\psi_2>_1|\psi_1>_2)|\psi_1>_3 \frac{1}{\sqrt{2}}(|\uparrow>_1|\downarrow>_2 - |\downarrow>_1|\uparrow>_2)|\downarrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 - |\psi_2>_1|\psi_1>_2)|\psi_1>_3 |\uparrow>_1|\uparrow>_2|\uparrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 - |\psi_2>_1|\psi_1>_2)|\psi_1>_3 |\uparrow>_1|\uparrow>_2|\downarrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 - |\psi_2>_1|\psi_1>_2)|\psi_1>_3 \frac{1}{\sqrt{2}}(|\uparrow>_1|\downarrow>_2 + |\downarrow>_1|\uparrow>_2)|\uparrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 - |\psi_2>_1|\psi_1>_2)|\psi_1>_3 \frac{1}{\sqrt{2}}(|\uparrow>_1|\downarrow>_2 + |\downarrow>_1|\uparrow>_2)|\downarrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 - |\psi_2>_1|\psi_1>_2)|\psi_1>_3 |\downarrow>_1|\downarrow>_2|\uparrow>_3$$
$$\frac{1}{\sqrt{2}}(|\psi_1>_1|\psi_2>_2 - |\psi_2>_1|\psi_1>_2)|\psi_1>_3 |\downarrow>_1|\downarrow>_2|\downarrow>_3$$

Therefore the degeneracy of the first excited state is 8.

(b) We again have three particles, 2 π^0 (particles 1 and 2) and one π^+ (particle 3). The pions are bosons so we need to make sure that the wavefunctions are symmetric under the interchange of identical pions. For the ground state we can put all pions in the n=1 level:

$$|\psi_1>_1|\psi_1>_2|\psi_1>_3$$

This is symmetric under interchange of particles 1 and 2, has energy $E = \alpha(2/m_{\pi_0} + 1/m_{\pi_+})$ and degeneracy 1.
Since the π_+ is heavier than the π_0, the n=2 level of the π_+ has lower energy than that of the π_0. The first excited state will thus have the π_+ in the n=2 level and the two π_0s in the n=1 level.

$$|\psi_1>_1|\psi_1>_2|\psi_2>_3$$

So the first excited state has energy $E = \alpha(2/m_{\pi_0} + 4/m_{\pi_+})$ and degeneracy 1.

(c) Two identical particles (one with spin up and one with spin down) can be put in each level of the square well. Therefore, of the five neutrons, two can be placed in n=1, two in n=2 and one in n=3. Of the three protons, two can be placed in n=1 and one in n=2. This gives a total energy for the ground state of
$E = \alpha((2\times1^2+2\times2^2+3^2)/m_n + (2\times1^2+2^2)/m_p) = \alpha(19/m_n + 6/m_p)$.

(d) All eleven pions can be put in the n=1 level, so the ground state energy is $E = \alpha(8/m_{\pi_0} + 3/m_{\pi_+})$.

NOTE: for this question, some students assumed that the neutron and proton were equal in mass (and also that the π_0 and π_+ had equal mass), and this is approximately true. This would change the degeneracies obtained above. I did not penalise students for making this assumption.

3. The totally antisymmetric state of three fermions is:

$$\begin{aligned}
|\Psi> &= \frac{1}{\sqrt{3!}}\begin{vmatrix} |\alpha>_1 & |\beta>_1 & |\gamma>_1 \\ |\alpha>_2 & |\beta>_2 & |\gamma>_2 \\ |\alpha>_3 & |\beta>_3 & |\gamma>_3 \end{vmatrix} \\
&= \frac{1}{\sqrt{6}}\left(|\alpha>_1 \begin{vmatrix} |\beta>_2 & |\gamma>_2 \\ |\beta>_3 & |\gamma>_3 \end{vmatrix} - |\beta>_1 \begin{vmatrix} |\alpha>_2 & |\gamma>_2 \\ |\alpha>_3 & |\gamma>_3 \end{vmatrix} + |\gamma>_1 \begin{vmatrix} |\alpha>_2 & |\beta>_2 \\ |\alpha>_3 & |\beta>_3 \end{vmatrix}\right) \\
&= \frac{1}{\sqrt{6}}(|\alpha>_1 |\beta>_2 |\gamma>_3 - |\alpha>_1 |\gamma>_2 |\beta>_3 - |\beta>_1 |\alpha>_2 |\gamma>_3 \\
&\qquad + |\beta>_1 |\gamma>_2 |\alpha>_3 + |\gamma>_1 |\alpha>_2 |\beta>_3 - |\gamma>_1 |\beta>_2 |\alpha>_3)
\end{aligned}$$

4. (a) Using the ultra-relativistic expression $E_F = \hbar c k_F = \hbar c(3\pi^2\rho)^{1/3}$ and approximating the kinetic energy as NE_F (correct up to a numerical factor), equation 10.51 of the text is replaced by:

$$\begin{aligned}
E_T = E_K + E_P &\approx N\hbar c(3\pi^2\rho)^{1/3} - \frac{3}{5}\frac{GM^2}{R} \\
&\approx N\hbar c\left(3\pi^2\frac{Zd}{AM_p}\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R} \qquad (\text{using } \rho \approx Zd/(AM_p)) \\
&\approx \frac{ZM}{AM_p}\hbar c\left(3\pi^2\frac{3ZM}{(4\pi R^3)AM_p}\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R} \\
&\qquad (\text{using } N \approx ZM/(AM_p) \text{ and } d = M/(4\pi R^3/3)) \\
&= \frac{bM^{4/3}}{R} - \frac{3}{5}\frac{GM^2}{R} \qquad \text{where } b = \left(\frac{3}{2}\right)^{2/3}\hbar c\frac{\pi^{1/3}}{M_p^{4/3}}\left(\frac{Z}{A}\right)^{1/3}
\end{aligned}$$

In fact there is a mistake in the statement of this question: stability is NOT associated with $E_T \leq 0$. If $bM^{4/3} \leq \frac{3}{5}GM^2$ then $E_T \sim -\frac{1}{R}$ and thus the star will collapse (minimum energy at $R = 0$). In this case $E_T \leq 0$ and the star is certainly not stable. Conversely, if $bM^{4/3} \geq \frac{3}{5}GM^2$ then $E_T \sim \frac{1}{R}$ and the star will expand to larger values of R until the electrons become non-relativistic. In this case the white dwarf will be stable. So stability requires:

$$\begin{aligned} bM^{4/3} &\geq \frac{3}{5}GM^2 \\ M^{2/3} &\leq \frac{5b}{3G} \\ M &\leq \left(\frac{5b}{3G}\right)^{3/2} \end{aligned}$$

Therefore the critical mass, above which the star is unstable, is $M_c = \left(\frac{5b}{3G}\right)^{3/2} = \frac{5^{3/2}\pi^{1/2}}{2 \times 3^{1/2}} \left(\frac{\hbar c}{G}\right)^{3/2} \left(\frac{Z}{AM_p}\right)^2$ (this is known as the Chandrasekhar limit).

(b) When a white dwarf star collapses, inverse beta decay can convert the electrons and protons to neutrons, leaving behind a neutron star. The neutrons then form a fermi gas. The calculation for Chandrasekhar limit is then identical to part (a), except now $Z = 1$ (since trivially there is one neutron per neutron "nucleus"), $A = 1$ and $M_p \to M_n$, so $M_c = \frac{5^{3/2}\pi^{1/2}}{2 \times 3^{1/2}} \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{M_n^2}$.

5. To calculate the integrals, note the following standard integral:

$$\int_0^\infty r^n e^{-\kappa r} dr = \frac{n!}{\kappa^{n+1}}$$

$$
\begin{aligned}
<\phi|-\frac{\hbar^2}{2m}\nabla_1^2|\phi> &= -\frac{\hbar^2}{2m}\frac{1}{\pi^2}\Big(\frac{\lambda}{a_0}\Big)^6\int d^3r_1\int d^3r_2 e^{-\lambda(r_1+r_2)/a_0}\Big(\tfrac{1}{r_1^2}\tfrac{\partial}{\partial r_1}(r_1^2\tfrac{\partial}{\partial r_1}e^{-\lambda(r_1+r_2)/a_0})\Big)\\
&= -\frac{\hbar^2}{2m}\frac{1}{\pi^2}\Big(\frac{\lambda}{a_0}\Big)^6\int d^3r_1\int d^3r_2 e^{-\lambda(r_1+r_2)/a_0}\Big(\tfrac{1}{r_1^2}\tfrac{\partial}{\partial r_1}(r_1^2(\tfrac{-\lambda}{a_0})e^{-\lambda(r_1+r_2)/a_0})\Big)\\
&= -\frac{\hbar^2}{2m}\frac{1}{\pi^2}\Big(\frac{\lambda}{a_0}\Big)^6(4\pi)^2\int_0^\infty dr_1 r_1^2\int_0^\infty dr_2 r_2^2(\tfrac{1}{r_1^2}(2r_1(\tfrac{-\lambda}{a_0})\\
&\qquad +r_1^2(\tfrac{-\lambda}{a_0})^2)e^{-2\lambda(r_1+r_2)/a_0}\\
&= -\frac{8\hbar^2}{m}\Big(\frac{\lambda}{a_0}\Big)^6\int_0^\infty dr_1(2r_1(\tfrac{-\lambda}{a_0})+r_1^2(\tfrac{-\lambda}{a_0})^2)e^{-2\lambda r_1/a_0}\int_0^\infty dr_2 r_2^2 e^{-2\lambda r_2/a_0}\\
&= -\frac{8\hbar^2}{m}\Big(\frac{\lambda}{a_0}\Big)^6(-(\tfrac{a_0}{2\lambda})-(\tfrac{a_0}{2\lambda}))\big(2!(a_0/2\lambda)^3\big)\\
&= \frac{\hbar^2}{2m}\Big(\frac{\lambda}{a_0}\Big)^2\\
&= \frac{e^2}{4\pi\epsilon_0}\frac{\lambda^2}{2a_0}
\end{aligned}
$$

$$
\begin{aligned}
<\phi|\frac{1}{r_1}|\phi> &= \frac{1}{\pi^2}\Big(\frac{\lambda}{a_0}\Big)^6\int d^3r_1\int d^3r_2\frac{1}{r_1}e^{-2\lambda(r_1+r_2)/a_0}\\
&= \frac{1}{\pi^2}\Big(\frac{\lambda}{a_0}\Big)^6(4\pi)^2\int_0^\infty dr_1 r_1^2\int_0^\infty dr_2 r_2^2\frac{1}{r_1}e^{-2\lambda(r_1+r_2)/a_0}\\
&= 16\Big(\frac{\lambda}{a_0}\Big)^6\int_0^\infty dr_1 r_1 e^{-2\lambda r_1/a_0}\int_0^\infty dr_2 r_2^2 e^{-2\lambda r_2/a_0}\\
&= 16\Big(\frac{\lambda}{a_0}\Big)^6\Big(\frac{a_0}{2\lambda}\Big)^2 2!\Big(\frac{a_0}{2\lambda}\Big)^3\\
&= \frac{\lambda}{a_0}
\end{aligned}
$$

By symmetry, we have the equivalent expressions for r_2:

$$
\begin{aligned}
<\phi|-\frac{\hbar^2}{2m}\nabla_2^2|\phi> &= \frac{e^2}{4\pi\epsilon_0}\frac{\lambda^2}{2a_0}\\
<\phi|\frac{1}{r_2}|\phi> &= \frac{\lambda}{a_0}
\end{aligned}
$$

Finally, as is calculated in detail in equations 10.74 to 10.81 of the text (now with λ replacing Z in the calculation):

$$
<\phi|\frac{e^2}{4\pi\epsilon_0 r_{12}}|\phi> = \frac{5e^2\lambda}{8(4\pi\epsilon_0)a_0}
$$

So:

$$
\begin{aligned}
<\phi|H|\phi> &= <\phi|-\frac{\hbar^2}{2m}\nabla_1^2|\phi> + <\phi|-\frac{\hbar^2}{2m}\nabla_2^2|\phi> - \frac{Ze^2}{4\pi\epsilon_0} <\phi|\frac{1}{r_1}|\phi> \\
&\qquad - \frac{Ze^2}{4\pi\epsilon_0} <\phi|\frac{1}{r_2}|\phi> + <\phi|\frac{e^2}{4\pi\epsilon_0 r_{12}}|\phi> \\
&= \frac{e^2}{4\pi\epsilon_0}\frac{\lambda^2}{2a_0} + \frac{e^2}{4\pi\epsilon_0}\frac{\lambda^2}{2a_0} - \frac{Ze^2}{4\pi\epsilon_0}\frac{\lambda}{a_0} - \frac{Ze^2}{4\pi\epsilon_0}\frac{\lambda}{a_0} + \frac{5e^2\lambda}{8(4\pi\epsilon_0)a_0} \\
&= \left(\lambda^2 - 2Z\lambda + \frac{5\lambda}{8}\right)\frac{e^2}{(4\pi\epsilon_0)a_0}
\end{aligned}
$$