Physics 137B (Professor Shapiro) Spring 2010

Homework 4 Solutions

1. The relativistic correction to the Hamiltonian was derived in class to be

$$H_1 = -\frac{p^4}{8m_e^3c^2}$$

Therefore the first order correction to the energy is

$$E_0^{(1)} = -\frac{\langle 0|\, p^4\, |0\rangle}{8m^3c^2} = -\frac{(\langle 0|\, p^2)(p^2\, |0\rangle)}{8m^3c^2}$$

Writing the momentum in terms of the operators a and $a^\dagger$ we find

$$p^2 = -\frac{\hbar m\omega}{2}(a^\dagger - a)^2 = -\frac{\hbar m\omega}{2}({a^\dagger}^2 - aa^\dagger - a^\dagger a + a^2)$$

Thus

$$p^2\, |0\rangle = -\frac{\hbar m\omega}{2}\left(\sqrt{1}\sqrt{2}\, |2\rangle - \sqrt{1}\sqrt{1}\, |0\rangle\right) = -\frac{\hbar m\omega}{2}\left(\sqrt{2}\, |2\rangle - |0\rangle\right)$$

Using the orthogonality of the wavefunctions, we therefore find

$$E_0^{(1)} = -\frac{1}{8m^3c^2}\frac{\hbar^2 m^2\omega^2}{4}(2+1) = -\frac{3(\hbar\omega)^2}{32mc^2}$$

2. Using the formulae from the problem, we find for a B field in the z direction

$$H' = -\frac{eB}{2m_e c}L_z$$

Since $[H', L_z] = 0$, the basis $|\ell\ m\rangle$ is the appropriate basis (the perturbation already is diagonal in this basis so we don't need to do a change of basis)

(a) The first order correction to the energy is

$$E^{(1)}_{\ell\, m} = -\frac{eB}{2m_e c} \langle \ell\; m | \, L_z \, | \ell\; m \rangle = -\frac{eB}{2m_e c} m$$

which breaks the degeneracy

(b) If we ignore the $L \cdot S$ coupling for now, then the argument from part a still holds. For the P-wave state of hydrogen ($\ell = 1$) the energy splits into 3 levels with energies 0 and $\pm\frac{eB}{2m_e c}$

3. The spin-orbit coupling is of the form $\vec{L} \cdot \vec{S}$. To see if $\vec{L}$ is conserved, we take the commutator of each component with H'. For example, the x compoent has a commutator:

$$[L_x, H'] = [L_x, L_x]S_x + [L_x, L_y]S_y + [L_x, L_z]S_z = i\hbar(L_z S_y - L_y S_z) \neq 0$$

Similarly, $[L_y, H'] \neq 0$ and $[L_z, H'] \neq 0$. The same argument holds for $\vec{S}$:

$$[S_x, H'] = [S_x, S_x]L_x + [S_x, S_y]L_y + [S_x, S_z]L_z = i\hbar(S_z L_y - S_y L_z) \neq 0$$

For the case of $J = L + S$ however

$$\begin{aligned} [L_x + S_x, H'] &= [L_x, L_x]S_x + [L_x, L_y]S_y + [L_x, L_z]S_z + [S_x, S_x]L_x + [S_x, S_y]L_y + [S_x, S_z]L \\ &= i\hbar(L_z S_y - L_y S_z + S_z L_y - S_y L_z) = 0 \end{aligned}$$

Similarly, the other 2 components vanish.

4. The expression for hyperfine splitting in hydrogen is:

$$\Delta E_{hf} = \frac{\mu_0 \hbar^2 e^2 g_e g_p}{6\pi a^3 m_e m_p}$$

The hyperfine splitting for hydrogen has a numeric value of 5.9×10^{-6} eV. The things that are different for the other systems listed in this problem are:

- The masses of the two particles change
- If the two particles are close together in mass, we should use the reduced mass to get the hydrogenlike energies

- The g-factor, which is 5.5857 for the proton, is 2 for the muon and for the electron.

In the case of positronium, the reduced mass is $m_e^2/(2m_e) = m_e/2$ so the hyperfine splitting is predicted to be:

$$\Delta E_{hf}(positronium) = \frac{m_p}{m_e}\frac{g_e}{g_p}\left(\frac{m_e/2}{m_e m_p/(m_p+m_e)}\right)^3 \Delta E_{hf}(hydrogen) = 4.8{\times}10^{-4}\ \text{eV}$$

Similarly, for muonium we get

$$\Delta E_{hf}(muonium) = \frac{m_p}{m_\mu}\frac{g_\mu}{g_p}\left(\frac{m_e m_\mu/(m_\mu+m_e)}{m_e m_p/(m_p+m_e)}\right)^3 \Delta E_{hf}(H) = 1.8{\times}10^{-5}\ \text{eV}$$

And for muonic hydrogen we get

$$\Delta E_{hf}(muonic\ hydrogen) = \frac{m_e}{m_m u}\frac{g_e}{g_\mu}\left(\frac{m_\mu m_p/(m_\mu+m_p)}{m_e m_p/(m_p+m_e)}\right)^3 \Delta E_{hf}(H) = 0.18\ \text{eV}$$

5. If we compare the hyperfine splitting in this systen to that of hydrogen, the differences are that the spin of the nucleus is 1 rather than $\frac{1}{2}$ and that the g factor is different for the deuteron is different from the proton. The spin of the deuteron nucleus is 1 and the spin of the electron is $\frac{1}{2}$. Therefore, the total spin of the system can be $\frac{3}{2}$ or $\frac{1}{2}$. Using $S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot S_2$ we can solve for $\vec{S}_1 \cdot S_2$. If $S = \frac{3}{2}$, we get $\hbar^2$ and if $S = \frac{1}{2}$ we get $-2\hbar^2$. Thus the splitting has a factor of $3\hbar^2$ rather than the $2\hbar^2$ we get for hydrogen. Using the same method as in the previous problem

$$\Delta E_{hf}(deuterium) = \frac{g_d}{g_p}\frac{m_p}{m_d}\frac{3}{2}\Delta E_{hf}(hydrogen) = 1.6 \times 10^{-6}\ \text{eV}$$