Physics 137B (Professor Shapiro) Spring 2010

GSI: Tom Griffin

Homework 1 Solutions

1. The hydrogenic atom involves a particle of charge $-e$ surrounding a nucleus of charge Ze. The Hamiltonian is given by (Equation 7.92 of text):

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{Ze^2}{(4\pi\epsilon_0)r}$$

where $\mu = \dfrac{mM}{m+M}$ is the reduced mass of the system. This Hamiltonian has ground state energy $E = -\dfrac{\mu}{2\hbar^2}(\dfrac{Ze^2}{4\pi\epsilon_0})^2$ and effective Bohr radius of $a = \dfrac{(4\pi\epsilon_0)\hbar^2}{Z\mu e^2}$ (see equations 7.114 and 7.151 of text). This result can then be applied to each case. In the following, $E_0 = -13.6eV$ denotes the hydrogen atom ground state energy and $a_0 = 5.29 \times 10^{-11}m$ denotes the hydrogen atom Bohr radius.

   (a) $Z = 1, m = m_e, M = m_{deuteron}$
   $E \approx E_0, \quad a \approx a_0$

   (b) $Z = 2, m = m_e, M = m_{He}$
   $E \approx 4E_0, \quad a \approx a_0/2$

   (c) $Z = 1, m = m_e, M = m_e$
   $\mu = m_e/2$
   $E \approx E_0/2, \quad a \approx 2a_0$

   (d) $Z = 1, m = m_\mu = 207 \times m_e, M = m_p$
   $\mu \approx 186 \times m_e$
   $E \approx 186E_0, \quad a \approx a_0/186$

(e) The Hamiltonian is now: $H = \frac{\mathbf{p}^2}{2\mu} - \frac{Gm_N^2}{r}$ with $\mu = m_N/2$ and so replacing $\frac{Ze^2}{(4\pi\epsilon_0)}$ by Gm_N^2 we have:

$$E = -\frac{\mu}{2\hbar^2}(Gm_N^2)^2 \approx 8.12 \times 10^{-69} eV,$$

$$a = \frac{\hbar^2}{G\mu m_N^2} \approx 7.16 \times 10^{22} m$$

2. The classically forbidden region occurs where the energy is lower than the potential energy. For a hydrogen atom the potential is $V = -\frac{e^2}{(4\pi\epsilon_0)r}$ and the ground state energy is $E_0 = -\frac{\mu}{2\hbar^2}(\frac{e^2}{4\pi\epsilon_0})^2 = -13.6eV$.

(a) Classically forbidden region is where:

$$\begin{aligned} E_0 &< V \\ E_0 &< -\frac{e^2}{(4\pi\epsilon_0)r} \\ r &> -\frac{e^2}{(4\pi\epsilon_0)E_0} = \frac{2\hbar^2(4\pi\epsilon_0)}{\mu e^2} = 2a_\mu \end{aligned}$$

(b) The probability of finding an electron in the forbidden region is:

$$
\begin{aligned}
P &= \int_{\text{forbidden region}} d^3\mathbf{r}|\psi_{100}(\mathbf{r})|^2 \\
&= \int_{2a_\mu}^{\infty} drr^2 \int d\Omega|\psi_{100}(\mathbf{r})|^2 \\
&= \int_{2a_\mu}^{\infty} drr^2 \int d\Omega|(\frac{1}{\sqrt{4\pi}})2(\frac{1}{a_\mu})^{3/2}e^{-r/a_\mu}|^2 \\
&= (\frac{4}{a_\mu^3})\int_{2a_\mu}^{\infty} drr^2 e^{-2r/a_\mu} \\
&= (\frac{4}{a_\mu^3})\left[e^{-2r/a_\mu}(-a_\mu r^2/2 - a_\mu^2 r/2 - a_\mu^3/4)\right]_{r=2a_\mu}^{r=\infty} \\
&= (\frac{4}{a_\mu^3})e^{-4}(2a_\mu^3 + a_\mu^3 + a_\mu^3/4) \\
&= 13e^{-4} \\
&\approx 0.24
\end{aligned}
$$

3. (a) The parity operator acts on the energy eigenstates by (equation 7.149 in text):

$$\mathcal{P}\psi_{nlm}(\mathbf{r}) = (-1)^l\psi_{nlm}(\mathbf{r}).$$

Therefore, for the state $\Psi(\mathbf{r}, t=0) = \frac{1}{\sqrt{14}}[2\psi_{100}(\mathbf{r}) - 3\psi_{200}(\mathbf{r}) + \psi_{322}(\mathbf{r})]$ we have:

$$\mathcal{P}\Psi(\mathbf{r}, t=0) = \Psi(\mathbf{r}, t=0).$$

So $\Psi(\mathbf{r}, t=0)$ is an eigenstate of parity, with parity eigenvalue $+1$.

(b) The probabilities are given by:

$$
\begin{aligned}
P_{100} &= |<\psi_{100}|\Psi(\mathbf{r}, t=0)>|^2 = |\frac{2}{\sqrt{14}}|^2 = 2/7 \\
P_{200} &= |<\psi_{200}|\Psi(\mathbf{r}, t=0)>|^2 = |\frac{-3}{\sqrt{14}}|^2 = 9/14 \\
P_{322} &= |<\psi_{322}|\Psi(\mathbf{r}, t=0)>|^2 = |\frac{1}{\sqrt{14}}|^2 = 1/14 \\
P_{\text{any other energy eigenstate}} &= |<\psi_{nlm}|\Psi(\mathbf{r}, t=0)>|^2 = 0
\end{aligned}
$$

(c) The expectation values are:

$$\begin{aligned}
< E > &= < \Psi(\mathbf{r}, t=0)|H|\Psi(\mathbf{r}, t=0) > \\
&= [(2/7)(1) + (9/14)(1/2^2) + 1/14(1/3^2)]E_0 \\
&= 229E_0/504 \\
< \mathbf{L^2} > &= < \Psi(\mathbf{r}, t=0)|\mathbf{L^2}|\Psi(\mathbf{r}, t=0) > \\
&= [(2/7)(0) + (9/14)(0) + 1/14(2(2+1))]\hbar^2 \\
&= 3\hbar^2/7 \\
< L_z > &= < \Psi(\mathbf{r}, t=0)|L_z|\Psi(\mathbf{r}, t=0) > \\
&= [(2/7)(0) + (9/14)(0) + 1/14(2)]\hbar \\
&= \hbar/7
\end{aligned}$$

4. The silver atoms enter the magnetic field with a velocity in the x direction of $v_x = (\frac{3kT}{M})^{1/2}$. The velocity in the x-direction does not change so the silver atoms are in the magnetic field for a time $t = \frac{L}{v_x} = L(\frac{M}{3kT})^{1/2}$. During this time they experience a force in the z-direction of $F_z = \mathcal{M}_z \frac{\partial B_z}{\partial z}$ which means that the silver atoms acquire a velocity in the z-direction of

$$v_z = \frac{F_z t}{M} = \frac{L\mathcal{M}_z \frac{\partial B_z}{\partial z}}{(3kTM)^{1/2}}$$

and travel a distance in the z-direction of

$$z_1 = \frac{1}{2}(\frac{F_z}{M})t^2 = \frac{1}{2}\frac{L^2\mathcal{M}_z \frac{\partial B_z}{\partial z}}{3kT}.$$

Before hitting the screen, the atoms then continue traveling for a time l/v_x which causes them to travel an additional distance in the z-direction of:

$$z_2 = v_z l/v_x = \frac{Ll\mathcal{M}_z \frac{\partial B_z}{\partial z}}{3kT}.$$

So the total distance traveled in the z-direction is:

$$z = z_1 + z_2 = \frac{L(l + L/2)\mathcal{M}_z \frac{\partial B_z}{\partial z}}{3kT} = \pm\frac{L(l + L/2)\mu_B \frac{\partial B_z}{\partial z}}{3kT} = \pm 3.9\text{cm}$$

which gives a maximum separation of 7.8cm.

5. Parts (a) to (g) are straightforward to derive using basic matrix algebra and the matrices:

$$\sigma_{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_{\mathbf{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_{\mathbf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Part (h) is a little more difficult and can be computed as follows. Note that by combining the results from parts (a), (b) and (c), we have the relation:

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

This then implies:

$$\begin{aligned}
(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) &= \sigma_i A_i \sigma_j B_j \\
&= \sigma_i \sigma_j A_i B_j \\
&= (\delta_{ij} + i\epsilon_{ijk}\sigma_k) A_i B_j \\
&= A_i B_i + i\epsilon_{ijk}\sigma_k A_i B_j \\
&= \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}$$